Achieve, Inc.

MAP Mathematics Expectations

Geometry

K.8 Create, explore and describe shapes.

K.8a Identify common shapes such as rectangle, circle, triangle, and square.

- Draw a variety of triangles (equilateral, right, isosceles, scalene) in a variety of positions.
- Draw squares and rectangles of different proportions (tall, squat, square-like) both horizontally and vertically positioned. Recognize tipped squares and rectangles.

  Note: Squares are a type of rectangle.

  Note: *Drawing* of tipped rectangles is typically too hard for kindergarten.
- Describe attributes of common shapes (e.g., number of sides and corners).

K.8b Use geometric tiles and blocks to assemble compound shapes.

- Assemble rectangles from two congruent right triangular tiles.
- Explore two-dimensional symmetry using matching tiles.

K.8c Recognize and use words that describe spatial relationships such as *above, below, inside, outside, touching, next to, far apart.*

Geometry

1.6 Recognize, describe, and draw geometric figures.

1.6a Identify and draw two-dimensional figures.

- Include trapezoids, equilateral triangles, isosceles triangle, parallelograms, quadrilaterals.

  Note: Be sure to include a robust variety of triangles as examples, especially ones that are very clearly not equilateral or isosceles.

- Describe attributes of two-dimensional shapes (e.g., number of sides and corners).

1.6b Identify and name three dimensional figures.

- Include spheres, cones, prisms, pyramids, cubes, rectangular solids.
- Identify two-dimensional shapes as faces of three-dimensional figures.

1.6c Sort geometric objects by shape and size.

- Recognize the attributes that determined a particular sorting of objects and use them to extend the sorting.

  Example: Various L-shaped figures constructed from cubes are sorted by the total number of cubes in each. Recognize this pattern, then sort additional figures to extend the pattern.

- Explore simultaneous independent attributes.

  Example: Sort triangular tiles according to the four combinations of two attributes such as right angle and equal sides.

1.7 Rotate, invert, and combine geometric tiles and solids.

1.7a Describe and draw shapes resulting from rotation and flips of simple two-dimensional figures.

- Identify the same (congruent) two-dimensional shapes in various orientations and move one on top of the other to show that they are indeed identical.
- Extend sequences that show rotations of simple shapes.

1.7b Identify symmetrical shapes created by rotation and reflection.

1.7c Use geometric tiles and cubes to assemble and disassemble compound figures.

- Count characteristic attributes (lines, faces, edges) before and after assembly.

  Examples: Add two right triangles to a trapezoid to make a rectangle; create a hexagon from six equilateral triangles; combine two pyramids to make a cube.

Geometry

2.9 Recognize, classify, and transform geometric figures in two and three dimensions.

2.9a Identify, describe, and compare common geometric shapes in two and three dimensions.

- Define a general triangle and identify isosceles, equilateral, right, and obtuse special cases.

  Note: The goal of naming triangles is not the names themselves but to focus on important differences. Triangles (or quadrilaterals) are not all alike, and it is their differences that give them distinctive mathematical features.

- Identify various quadrilaterals (rectangles, trapezoids, parallelograms, squares) as well as pentagons and hexagons.

  Note: In this grade *parallel* is used informally and intuitively; it receives more careful treatment at a later grade.

  Note: A *square* is a special kind of rectangle (since it has four sides and four right angles); a *rectangle* is a special kind of parallelogram (since it has four sides and two pairs of parallel sides); and a *parallelogram* is a special kind of *trapezoid* (since it has four sides and at least one pair of parallel sides); and a trapezoid is a polygon (since it is a figure formed of several straight sides). So for example, contrary to informal usage, in mathematics a square is a trapezoid.

- Understand the terms *perimeter* and *circumference.*

  Note: The primary meaning of both terms is the outer boundary of a two-dimensional figure; circumference is use principally in reference to circles. A secondary meaning for both is the length of the outer boundary. Which meaning is intended needs to be determined from context.

- Distinguish circles from ovals; recognize the circumference, diameter, and radius of a circle.
- In three dimensions, identify spheres, cones, cylinders, triangular and rectangular prisms.

2.9b Describe common geometric attributes of familiar plane and solid objects.

- Common geometric attributes include position, shape, size, roundness, and numbers of corners, edges, and faces.
- Distinguish between geometric attributes and other characteristics such as weight, color, or construction material.
- Distinguish between lines and curves, and between flat and curved surfaces.

2.9c Rotate, flip, and fold shapes to explore the effect of transformations.

- Use paper folding to find lines of symmetry.
- Recognize congruent shapes.
- Identify shapes that have been moved (flipped, slid, rotated), enlarged, or reduced.

2.10 Understand and interpret rectangular arrays as a model of multiplication.

2.10a Create square cells from segments of the discrete number line used as sides of a rectangle.

- Match cells to discrete objects lined up in regular rows of the same length.

2.10b Understand rectangular arrays as instances of repeated addition.

Geometry

3.8 Recognize basic elements of geometric figures and use them to describe shapes.

3.8a Identify points, rays, line segments, lines, planes in both mathematical and everyday settings.

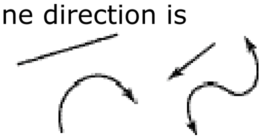
- A *line* is a straight path traced by a moving point having no breadth nor end in either direction.

  Examples: Each figure on the left above represents a line; the arrows indicate that the lines keep going in the indicated directions without end. The number line with both positive and negative numbers is a line.

- A part of a line that starts at one point and ends at another is called a *line segment*. Line segments are drawn without arrows on either end because line segments end at points.

  Examples: The figure in the center above is a line segment. The edges of a desk or door or piece of paper are everyday examples of line segments.

- Part of a line that starts at one point and goes on forever in one direction is called a *ray*.

  ***Examples*:** The figure above on the right is a ray. The positive number line (to the right of 0) is a ray. On the other hand, none of the four examples at the right are lines:

  Caution: Not all sources distinguish carefully among the terms *line*, *segment*, or *ray*, nor do all sources employ the convention of arrowheads in exactly the manner described above. Often context is the best guide to distinguish among these terms.

- Know that a *plane* is a flat surface without thickness that extends indefinitely in every direction.

  Examples: Everyday examples that illustrate a part of a plane are the flat surfaces of a floor, desk, windowpane, or book. Examples that are not part of a plane are the curved surfaces of a light bulb, a ball, or a tree.

3.8b Understand the meaning of parallel and perpendicular and use these terms to describe geometric figures.

- Lines and planes are called *parallel* if they do not meet no matter how far they are extended
- Lines and planes are called *perpendicular* if the corners formed when they meet are equal.
- Identify parallel and perpendicular edges and surfaces in everyday settings (e.g., the classroom).

  Examples: The lines on the page of a notebook are parallel, as are the covers of a closed book are parallel. Corners of books, walls, and rectangular desks are perpendicular, as are the top and side edges of a chalk board and a wall and a floor in a classroom.

- The corner where two perpendicular lines meet is called a *right angle*.

  Note: The general concept of "angle" is developed later; here the term is used merely as the name for this specific and common configuration.

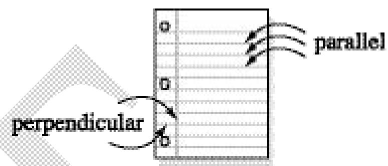
- Understand and use the terms *vertical* and *horizontal*.

- Recognize that vertical and horizontal lines or planes are perpendicular, but that perpendicular lines or planes are not necessarily vertical or horizontal.

3.8c Use terms such as line, plane, ray, line segment, parallel, perpendicular, and right angle correctly to describe everyday and geometric figures.

3.9 Identify and draw perpendicular and parallel lines and planes.

3.9a Draw perpendicular, parallel, and non-parallel line segments using rulers and squares.



- Recognize that lines that are parallel to perpendicular lines will themselves be perpendicular.

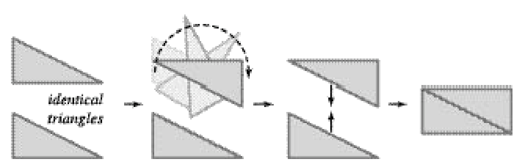
  Example: Fold a piece of paper in half from top to bottom, then fold it in half again from left to right. This will give two perpendicular fold lines and four right angles.

- Edges of a polygon are called parallel or perpendicular if they lie on parallel or perpendicular lines, respectively.
- Similarly, faces of a three-dimensional solid are called parallel or perpendicular if they lie in parallel or perpendicular planes, respectively.

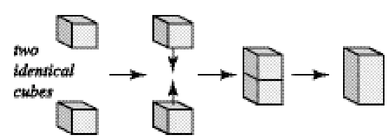
3.10 Explore and identify familiar two- and three-dimensional shapes.

3.10a Describe and classify plane figures and solid shapes according to the number and shape of faces, edges, and vertices.

- Plane figures include circles, triangles, squares, rectangles, other polygons); solid shapes include spheres, pyramids, cubes, and rectangular prisms.
- Recognize that the exact meaning of many geometric terms (e.g., rectangle, square, circle, and triangle) depends on context: sometimes they refer to the boundary of a region and sometimes to the region contained within the boundary.



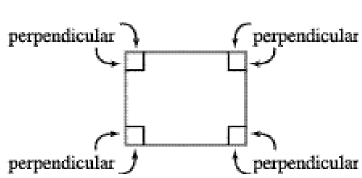
3.10b Know how to put shapes together and take them apart to form other shapes.

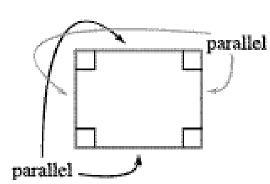


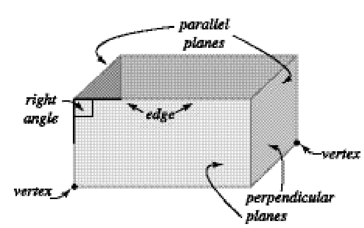
Examples: Two identical right triangles can be arranged to form a rectangle (see figure above). Two identical cubes can be arranged to form a rectangular prism (figure at right).

3.10c Identify edges, vertices (corners), perpendicular and parallel edges, and right angles in two-dimensional shapes.

Example: A rectangle has four pairs of perpendicular edges, two pairs of parallel edges, and four right angles.





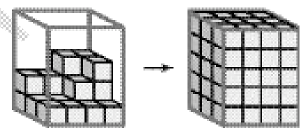


3.10d Identify right angles, edges, vertices, perpendicular and parallel planes in three-dimensional shapes.

3.11 Understand how to measure length, area, and volume.

3.11a Understand that measurements of length, area, and volume are based on standard units.

- Fundamental units are: a *unit interval* of length 1 unit, a *unit square* whose sides have length 1 unit, and a *unit cube* whose sides have length 1 unit.
- The *volume* of a rectangular prism is the number of unit cubes required to fill it exactly (with no space left over).

  Note: The common childhood experience of pouring water or sand offers a direct representation of volume.
- The *area* of a rectangle is the number of unit squares required to pave the rectangle--that is, to cover completely without any overlapping.

  Note: Area provides a critical venue for developing the conceptual underpinnings of multiplication.
- The *length* of a line segment is the number of unit inervals that are required to cover the segment exactly with nothing left over.

3.11b Know how to calculate the perimeter, area, and volume of shapes made from rectangles and rectangular prisms.

- The *perimeter* of a rectangle is the number of unit intervals that are required to enclose the rectangle.
- Measure and compare the areas of shapes using non-standard units (e.g., pieces in a set of pattern blocks).
- Recognize that the area of a rectangle is the product of the lengths of its base and height ($A = b \times h$), and that the volume of a rectangular prism is the product of the lengths of its base, width, and height ($V = b \times w \times h$).

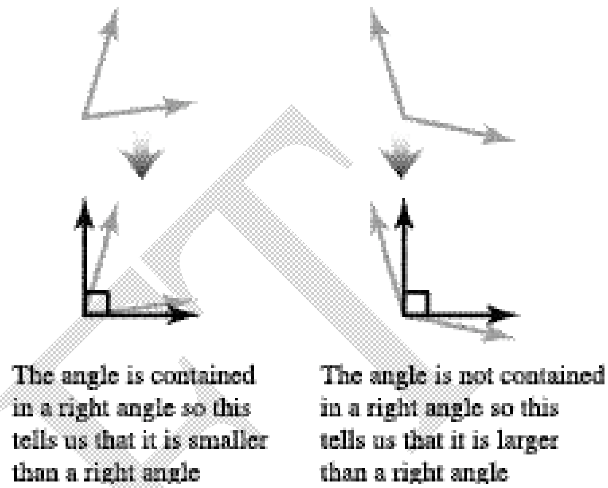
  Example: Build solids with unit cubes and use the formula for volume (V=*bwh)* to verify the count of unit cubes; make similar comparisons with rectilinear figures in the plane that are created from unit squares.
- Find the area of a complex figure by adding and subtracting areas.
- Compare rectangles of equal area and different perimeter, and also rectangles of equal perimeter and different area.
- Measure surface area of solids by covering each face with copies of a unit square, and then counting the total number of units.

Geometry

4.11 Understand and use the definitions of angle, polygon and circle.

4.11a An angle in a plane is a region between two rays that have a common starting point.

Note: According to this definition, a right angle (as determined by perpendicular rays) is indeed an angle.



4.11b If angle A is contained in another angle B, then angle B is said to be bigger than angle A.

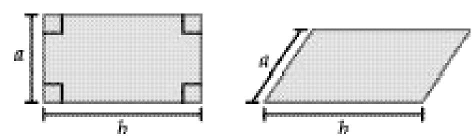
- The figure on the right illustrates how to determine whether an angle is larger than, smaller than, or close to a right angle.

Note: When two rays come from the same point (see figure at right) they divide the plane into two regions, giving two angles. Except where otherwise indicated, the angle is determined by the two rays is defined, by convention, as the smaller region.

- Understand that shapes such as triangles, squares, rectangles have angles.

Note: Technically, polygons do not contain rays, which are required for the definition of angles. Their sides are line segments of finite length. Nonetheless, if we imagine the sides extending indefinitely away from each corner, then each corner becomes an angle.

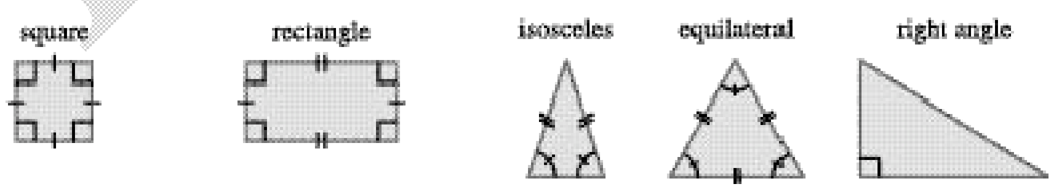
Example: Describe the difference between the two figures on the right:



- Identify *acute*, *obtuse*, and *right* angles.

4.11c Know and use the basic properties of squares, rectangles, and isosceles, equilateral, and right triangles.

- Identify *scalene*, *acute* and *obtuse* triangles.
- Know how to mark squares, rectangles, and triangles appropriately:



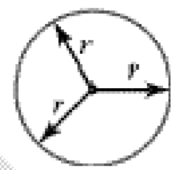
4.11d Know what a polygon is and be able to identify and draw some examples.

- A *polygon* is a figure that lies in a plane consisting of a finite number of line segments called *edge*s (or *sides*) with the properties that (a) each edge is joined to exactly two other edges at the end points; edges do not meet each other except at end points; and the edges enclose a single region.

  Example: The figures on the right are *not* polygons:

4.11e Know and use the basic properties of a circle.

circle



- A circle as the set of points in a plane that are at a fixed distance from a given point.
- Know that a circle is not a polygon.

Geometry

5.11 Measure angles in degrees and solve related problems.

5.11a Understand the definition of *degree* and be able to measure angles in degrees.

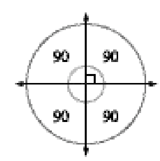
- A *degree* is one part of the circumference of a circle of radius 1 unit (a *unit* circle) that is divided into 360 equal parts. The measure of an angle in degrees is defined to be the number of degrees of the arc of the unit circle, centered at the vertex of the angle, that is intercepted by the angle.
- The measure of an angle in degrees can also be interpreted as the amount of counter-clockwise turning from one ray to the other.

    Note: Earlier (in Grade 4) the angle determined by two rays was defined to be *smaller* of the two options. For consistency, therefore, when an angle is measured by the amount of turning necessary to rotate one ray into another, it is important to start with the particular ray that will produce an angle measure no greater than 180°

- The symbol ° is an abbreviation for "degree" (e.g., 45 degrees = 45°).
- As a shorthand, angles are called equal if the measures of the angles are equal.

5.11b Know and use the measures of common angles.

- Recognize that angles on a straight line add up to 180° and that angles around a point add up to 360°. An angle of 180° is called a *straight* angle.
- A right angle is an angle of 90°. An acute angle is an angle of less than 90°, while an obtuse angle is an angle of more than 90°



    Note: Since a pair of perpendicular lines divides the plane into 4 equal angles, the measure of a right angle is 360°/4 = 90°.

5.11c Interpret and prepare circle graphs (pie charts).

5.12 Know how to do basic constructions using a straight edge and compass.

5.12a Basic constructions include (a) drop a perpendicular from a point to a line, (b) bisect an angle; (c) erect the perpendicular bisector of a line, and (d) construct a hexagon on a circle.

Note: A straightedge is a physical representation of a line, not a ruler which is used for measuring. The role of a straightedge in constructions is to draw lines through two points, just as the role of the compass is to draw a circle based on two points, the center and a point on the circumference.

Note: Students need extended practice with constructions since they embody the elements of geometry--lines and circles--independent of numbers and measurement. Since constructions are so central to Euclidean geometry, they are often called *Euclidean constructions*.

Note: These constructions are *basic* in the sense that other important constructions introduced in later grades (e.g., of an equilateral triangle given one side; of a square inscribed in a circle) build on them.

- Use informal arguments such as paper folding to verify the correctness of constructions.

5.13 Recognize and work with simple polyhedra.

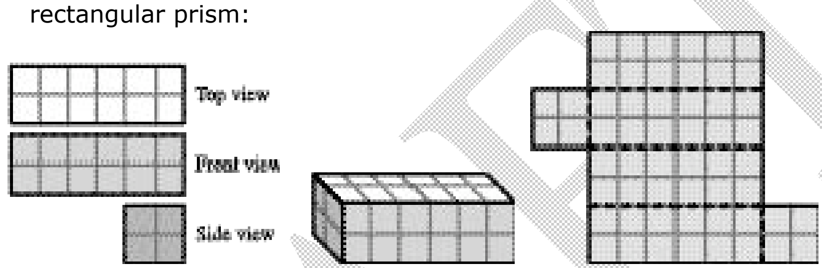
5.13a Represent and work with rectangular prisms by means of orthogonal views, projective views, and nets.

- A *net* is a flat (two-dimensional) pattern of faces nets that can be folded to form the surface of a solid.

  Note: Because a net represents the surface of a polyhedra spread out in two dimensions, the area of a net equals the surface area of the corresponding solid.

- Orthogonal views are from top, front and side; picture views are either projective or isometric; and nets are plane figures that can be folded to form the surface of the solid.

  Example: An orthogonal view (a), a projective view (b) and a net (c) of the same rectangular prism:



(a) (b) (c)

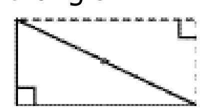
5.13b Recognize the five regular ("Platonic") solids.

- Count faces, edges, and vertices, and make a table with the results.

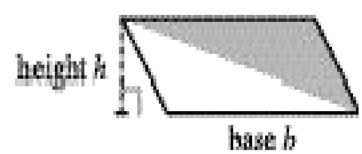
5.14 Find the area of shapes created out of triangles.

5.14a Understand, derive, and use the formula $A = \frac{1}{2}bh$ for the area of a triangle.

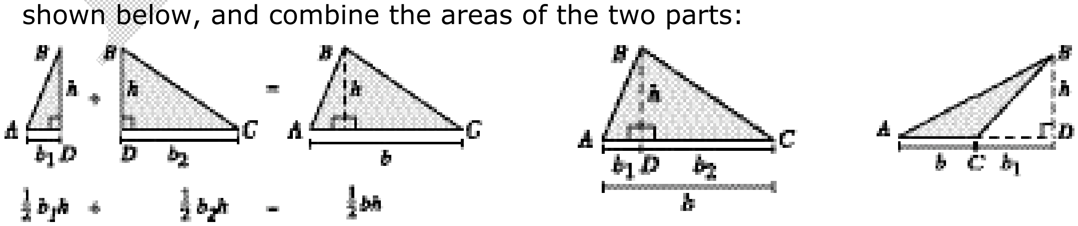
- Arrange two identical right triangles with base *b* and height *h* to form a rectangle whose area is *bh*. Since the area of each right triangle is half that of the rectangle, $A = \frac{1}{2}bh$.



- If triangle ABC is not a right triangle, then placing two copies together will form a parallelogram with base *b* and height *h*. This parallelogram can be transformed into a rectangle of area *bh* by moving a right triangle of height *h* from one side of the parallelogram to the other. So here too, $A = \frac{1}{2}bh$.



- Alternatively, to divide a general triangle ABC into two right triangles as shown below, and combine the areas of the two parts:



  Note: As the diagrams show, there are two cases to consider: For an acute triangle (where all angles are smaller than a right angle), the parts are added together. For an obtuse triangle (where one angle is larger than a right angle), one right triangle must be subtracted from the other.

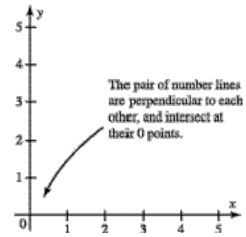
5.14b Find the area of a convex polygon by decomposing it into triangles.

- A polygon is called *convex* if a line segment joining any two points on the perimeter of the polygon will lie inside or on the polygon.
- Any convex polygon of *n* sides can be decomposed into ($n-2$) triangles.

5.14c Find the area of other geometric figures that can be paved by triangles.

5.15 Interpret and plot points on the coordinate plane.

5.15a Associate an ordered pair of numbers with a point in the first (upper right) quadrant and, conversely, any such point with an ordered pair of numbers.



- Positions on the coordinate plane are determined in relation to the coordinate axes, a pair of number lines that are placed perpendicular to each other so that the zero point of each coincides.

  Note: The coordinate plane is a two-dimensional extension of the number line and builds on extensive (but separate) prior work with the number line and with perpendicular lines.
- Recognize the similarity between locating points on the coordinate plane and locating positions on a map.
- Recognize and use the terms *vertical* and *horizontal*.

5.15b Identify characteristics of the set of points that define vertical and horizontal line segments.

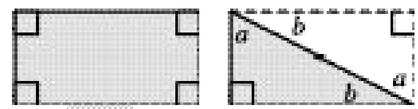
- Use subtraction of whole numbers, fractions, and decimals to find the length of vertical or horizontal line segments identified by the ordered pairs of its endpoints.

  Example: What is the length of the line segment determined by (3/5, 0) and (1.5, 0)?

Geometry

6.7 Understand and use basic properties of triangles and quadrilaterals.

6.7a Understand and use the angle properties of triangles and quadrilaterals.



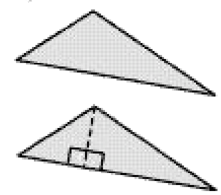
- The sum of angles in a right triangle is 180° since two identical right triangles form a rectangle,

  Note: By definition a rectangle has 4 right angles, so the sum of the angles of a rectangle is 4 x 90° = 360°. Each right triangle contains half 360°, or 180°.

  Note: Since one angle in a right triangle is 90°, the sum of the remaining two angles is also 90°.

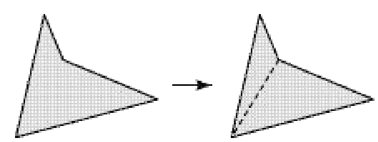
- Since the sum of angles in a parallelogram is also 360°, the sum of angles in any triangle is also 360°/2 = 180°.

  Note: Following the line of argument used in Grade 5 to find the area of a triangle, we note that (a) two identical copies of any triangle can be arranged to form a parallelogram, and (b) any parallelogram, can be transformed into a rectangle with the same angle sum by moving a triangle from one side of the parallelogram to the other.

  Note: Alternatively, following the secondary argument offered in Grade 5, one can drop a perpendicular to divide any triangle into two right triangles. The sum of the interior angles of each of these right triangles is 180°, but when put together they include two superfluous right angles. Subtracting these yields 180° + 180° – (90° + 90°) = 180° as the sum of the interior angles of any triangle.

- Since any quadrilateral can be divided into two triangles the sum of the angles in a quadrilateral is also 2 × 180° = 360°.



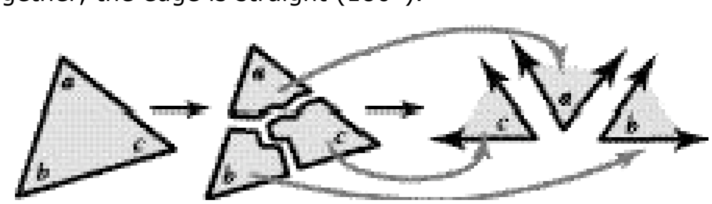
6.7b Use a protractor, ruler, square, and compass to draw triangles and quadrilaterals from data given in either numerical or geometric form.

- Draw a variety of triangles (right, isosceles, acute, obtuse) and quadrilaterals (squares, rectangles, parallelograms, and trapezoids) of different dimensions.
- Verify basic properties of triangles and quadrilaterals by direct measurement.

  Note: Verification by measurement requires many examples, especially some with relatively extreme or uncommon dimensions.

  Examples: In parallelograms, opposite sides and opposite angles are equal; in rectangles, diagonals are equal.

  Example: Cut any triangle out of paper and tear it into three parts so that each part contains one of the triangle's vertices. Notice that when the angles are placed together, the edge is straight (180°).



- Explore properties of triangles and quadrilaterals with dynamic geometry software.

Note: Is a parallelogram a trapezoid? It depends on the definition of trapezoid. If a trapezoid is defined as a quadrilateral with *at least* one pair of parallel edges, parallelograms become special cases of trapezoids. However, dictionaries usually define a trapezoid as a quadrilateral with *exactly one* pair of parallel edges, thereby distinguishing between parallelograms and trapezoids. Mathematicians generally prefer nested definitions as conditions become more or less restrictive. For example, all positive whole numbers are integers, all integers are rational numbers, and all rational numbers are real numbers. So in the world of mathematics, squares are rectangles, rectangles are parallelograms, and parallelograms are trapezoids.

6.8 Understand and use basic properties of angles, lines and triangles in the plane.

6.8a Understand the triangle inequality, verify it through measurement, and recognize when it can be useful to solve problems.

- In words, the longest side of a triangle is shorter than the sum of the other two sides. In symbols, if a, b, and c are three sides of a triangle with $a \le b \le c$, then $c < a + b$.

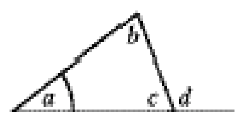
  Note: Since those sides of a triangle that are not the longest are by definition shorter than the longest, they too are obviously shorter than the sum of the other two. Thus the triangle inequality applies to all three sides: $a < b + c$, $b < a + c$, and $c < a + b$.

  Note: Although relatively simple, the triangle inequality is a deep and fundamental insight that recurs throughout advanced mathematics.

6.8b Know the definitions and properties of interior and exterior angles.

- Understand why each exterior angle of a triangle is equal to the sum of the opposite interior angles.

  Note: In the triangle shown, $a + b + c = 180°$ and also $d + c = 180°$ (since c and d are supplementary angles. Therefore $d = a + b$.



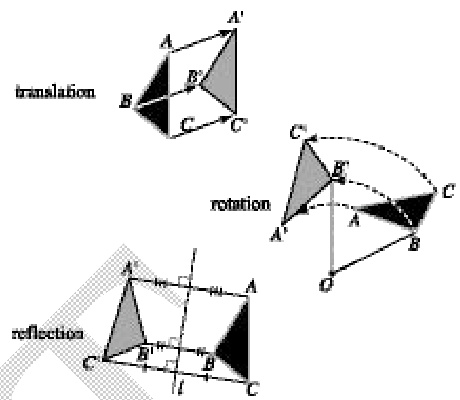
6.8c Understand why the sum of the interior angles of an n-sided convex polygon is $(n - 2) \times 180°$.

- Strategy: decompose an n-sided polygon into $n - 2$ triangles.

6.8d Understand why the sum of exterior angles of a convex polygon is 360°.

- A person walking around the perimeter would make one complete revolution. So the sum of exterior angles must be 360°.

  Note: Here is amore formal explanation for a pentagon**.** The sum of each adjacent interior angle (I) and exterior angle (E) is 180°. Since there are 5 such pairs of angles, the sum of all interior and exterior angles of a pentagon is $5 \times 180° = 900°$. As noted above, the sum of the interior angles is $(5-2) \times 180° = 540°$. Subtracting, we get $900° - 540° = 360°$ as the sum of the exterior angles.

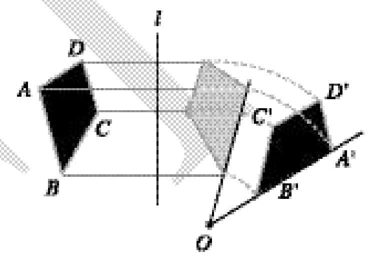


6.9 Understand and use the concepts of translation, rotation, reflection, and congruence in the plane.

6.9a Recognize that every rigid motion of a polygon in the plane can be created by some combination of translation, rotation, and reflection.

- Translation, rotation, and reflection move a polygonal figure in the plane from one position to another without changing its length or angle measurements.
- Explore the meaning of rotation, translation, and reflection through drawings and hands-on experiments.

  Example: The figure on the right illustrates how a rigid motion can be decomposed into a series of three steps: translation, reflection, and rotation.



6.9b Understand several different characterizations and examples of congruence.

- Two figures in the plane are called *congruent* if they have the same size and same shape.
- Two shapes are congruent if they can be made to coincide when superimposed by means of a rigid motion.
- Two polygons are congruent if they have the same number of sides and if their corresponding sides and angles are equal.

  Note: Historically--beginning with Euclid--congruence applied only to polygons, and used this as the definition. Indeed, the important properties of congruence are typically only about polygonal figures.
- Congruent figures in the plane are those that can by laid on top of one another by rotations, reflections, and translations.

  Note: Using rotations, reflections, and translations to define congruence gives precise meaning to the intuitive idea of congruence as "same size and same shape," thus permitting a precise definition of congruence for shapes other than polygons.

  Note: Technological aids (transparencies, dynamic geometry programs) help greatly in studying rigid motions.

6.9c Identify congruent polygonal figures.

- Understand why the two triangles formed by drawing a diagonal of a parallelogram are congruent.
- Understand why the two triangles formed by bisecting the vertex angle of an isosceles triangle are congruent.

6.10 Understand and use different kinds of symmetry in the plane.

6.10a Symmetries in the plane are actions that leave figures unchanged.

- Explore and explain the symmetry of geometric figures from the standpoint of rotations, reflections and translations.

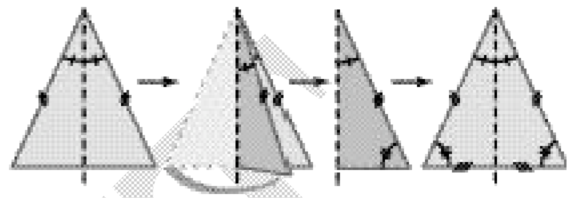
6.10b Identify and utilize bilateral and rotational symmetry in regular polygons.

- A *regular polygon* is a polygon whose sides and angles are all equal.
- *Bilateral symmetry* means there is a reflection that leaves everything unchanged.
- *Regular polygons* have rotational symmetries.

6.10c Identify and utilize translational symmetry in tessellations of the plane.

- Most tessellations have translation symmetry.

6.10d Use reflections to study isosceles triangles and isosceles trapezoids.



- Understand why the base angles of an isosceles triangles are equal
- Understand why the bisector of the angle opposite the base is the perpendicular bisector of the base.

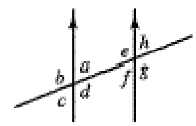
  Note: Draw an angle bisector on an isosceles triangle. Fold the drawing along the angle bisector (that is, reflect across the angle bisector). Then the base vertices collapse on each other: both angles are equal, thus the angle bisector also bisects the base.

Geometry

7.5 Understand angle properties of parallel lines in the plane.

7.5a Understand the definitions and properties of vertical, adjacent, complementary, supplementary, corresponding, and alternate interior angles.

- When a line intersects two parallel lines, angles *a* and *c* are *vertical*, *a* and *b* are *supplementary*, *a* and *h* are *corresponding* and *a* and *f* *alternate interior* angles°.



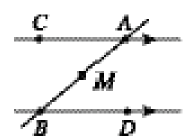
7.5b Understand why vertical angles are equal.

- Vertical angles are equal because a straight angle is 180° and each vertical angle is the supplement of a single angle. In the diagram above, $a + b = 180° = b + c$, so $a = c$.

7.5c Recognize that corresponding and alternate interior angles are equal.

- (a) If a line intersects two parallel lines, corresponding angles and alternate interior angles must be equal.

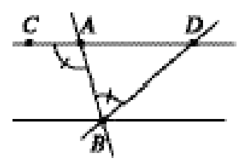
  Note: To see that alternate interior angles are equal, consider the diagram at the right where lines *CA* and *BD* are parallel. Let *M* be the midpoint of line segment *AB*. Now rotate the line *BD* 180° around the point *M.* Then the point *B* falls on point *A*, and the ray *BD* becomes the ray *AC* because the lines are parallel. Therefore ∠*MBD* falls on ∠*MAC*, which means that the two angles are equal.



  Note: In asserting that ray BD becomes ray AC, the preceding argument assumes that there can be no more than one line through point A that is parallel to the line through BD. This makes hidden use of Euclid's famous fifth postulate about parallel lines.

- (b) If a line intersects two other lines in such a way as to make the corresponding angles and alternate interior angles equal, then these two lines must be parallel.

  Note: To show this, suppose two lines make equal alternate interior angles, ∠*CAB* and ∠*ABD* as shown. Suppose that lines *CA* and *BD* were not parallel*.* Then the two lines must intersect at some point, say D (see diagram). Then points ABD are the vertices of a triangle. As shown in Grade 6, the exterior angle ∠*CAB* of this triangle equals the total of the two interior angles: ∠*CAB* =∠*ABD* +∠*ADB*. However, this contradicts the assumption that ∠*CAB* =∠*ABD*. So our supposition that CA and BD were not parallel must not be true. In other words, *AC* and line *BD* are parallel.



  Note: The second assertion is the *converse* of the first. Whereas (a) says that P implies Q, (b) says that Q implies P. Together they form an "if and only if" relationship: "If a line L intersects two other lines, corresponding and alternate interior angles are equal if and only if these two other lines are parallel."

7.5d Recognize the meanings and roles of assumption, conclusion, converse, and indirect argument.

- Propositions in geometry take the form "if P, then Q." P is the *assumption* (or *hypothesis*), Q is the *conclusion*.
- The proposition "if Q, then P" is the *converse* of "if P, then Q."

  Note: Sometimes the converse of a proposition is true, but usually it is not.

Examples: Statements (a) and (b) below are converses of each other, as are statements (c) and (d). Statements (a) and (b) are both true, as is statement (c). However, statement (d) is false.

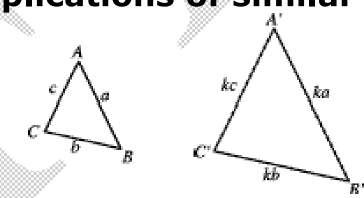
(a) If a parallelogram is a rectangle, then its diagonals are equal.
(b) If the diagonals of a parallelogram are equal then the parallelogram is a rectangle.
(c) If a quadrilateral is a rectangle, then its diagonals are equal.
(d) If the diagonals of a quadrilateral are equal, then the quadrilateral is a rectangle.

- An *indirect argument* is a method of reasoning that shows that a conclusion cannot be false rather than showing directly that it is true.

  Note: The proof given above that if alternate interior angles are equal, the lines creating them be parallel is an indirect argument. In particular, it is an *argument by contradiction*. Instead of showing directly that the assumption (equal alternate angles) implies the conclusion (parallel lines), it shows that if the lines are *not* parallel then the two angles would *not* be equal. Since we know that they are equal (by assumption), the lines cannot *not* be parallel. So they must be parallel. There is no other alternative. Arguments by contradiction are very common in mathematics, and that is one reason that this particular result is worth emphasizing.

7.6 Understand the definition, criteria, and applications of similar triangles.

7.6a Understand the definition of similarity for triangles.



- Informally, two triangles are *similar* if they have the same shape.
- Formally, two triangles are similar if their corresponding angles are equal.

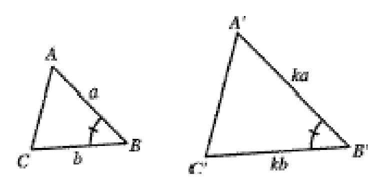
  Note: The common meaning of *similar* is having the same shape. It is easy to illustrate using the magnification feature of copiers and computers that for triangles, having the same angles yields the same shape.

  Caution: This is not true for quadrilaterals. All rectangles have equal corresponding angles (they are all 90°) but they are not all the same shape.

7.6b Recognize several criteria for similarity of triangles.

- Two triangles are similar if the ratios of the lengths of corresponding sides are equal (SSS criterion).

  Note*:* Verification should be empirical, using rulers and protractors.



- Two triangles are similar if the ratios of the lengths of two pairs of corresponding sides are equal and the corresponding angles between them are equal (SAS criterion).

  Example: If AB/A'B' = BC/B'C' and ∠B = ∠B', then triangle ABC is similar to triangle A'B'C'.

- Triangles are similar if two pairs of corresponding angles are equal (AA criterion).

  Note: The converses of all three of these criteria are all true: if two triangles are similar, they have the properties named in the criteria. Thus they could as well be stated as "if and only if" criteria.

7.6c Recognize that when a line is drawn inside a triangle parallel to one side, it forms a smaller triangle similar to the original one.

- Use equality of ratios of different line segments formed when a line inside a triangle is drawn parallel to one side.
- If points D and E in triangle ABC are on sides AB and AC, respectively, and

the line DE is parallel to BC, then AD:AM = AE:AC and AD:DB = AE:EC.

Note: The notation AD:AM stands for the ratio of the lengths of line segment AD to that of line segment AM. The expression AD:AM = AE:AC symbolizes the "equality of ratios." If AD is three times as long as AM, then AE will be three times as long as EC.

7.6d Use similar triangles to find the lengths of unknown line segments in a triangle.

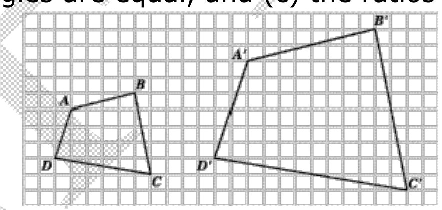
- Employ the notion of similarity to solve geometric problems where direct measurement is difficult or impossible.

  Note: The equality of ratios created by similar triangles is of crucial importance in the following study of the slope of a line.

7.7 Extend similarity to other polygons in the plane

7.7a Understand both informal and formal definitions of similarity for polygons in the plane.

- Informally, similar polygons are those that have the same shape.
- Formally, two polygons are similar if (a) that they have the same number of sides, (b) the measures of corresponding angles are equal, and (c) the ratios of the lengths of corresponding sides are equal.



  Note: In the diagram quadrilateral ABCD and A'B'C'D' are similar. This means that ∠A = ∠A', ∠B = ∠B', ∠C = ∠C', ∠D = ∠D' and that AB/A'B' = BC/B'C' = CD/C'D' = DA/D'A'.

  Note: The value of the common ratio of the sides of similar polygons is called the *scale factor*.

- Understand that congruent figures are similar.

  Note: By definition, congruent polygons are those that have the same shape and same size. So they satisfy the "same shape" criterion for similarity.

- Use examples to show that analogues of the SSS, SAS, and AA criteria for similarity of triangles do not work for polygons with more than three sides.

7.7b Understand and use the scale factor through which a polygon can be transformed into a different figure that is similar to the original.

- Relate the scale factor to the scale on maps and in scale model drawings.

  Note: In Grade 8 similarity will be defined precisely for all figures by means of dilations.

7.8 Understand the definition of slope, how to calculate it, and use slope to solve problems.

7.8a Understand and calculate the slope of a line in a coordinate plane.

- The slope of a line is the ratio of the lengths of vertical and horizontal line segments--the "rise" and "run"--that together with the line form a right triangle in a coordinate plane.
- Calculate the slope a line in a coordinate plane.

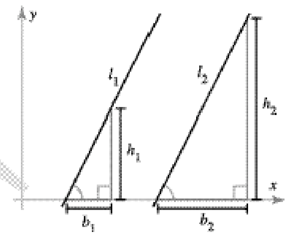
7.8b Understand the relation of slope to parallel and perpendicular lines in a coordinate plane.

- The calculated slope of a line in a coordinate plane is the same no matter which two points one uses to perform the calculation.

Note: This can be verified using similar triangles.

- Two lines in a coordinate plane are parallel if and only if they have the same slope.

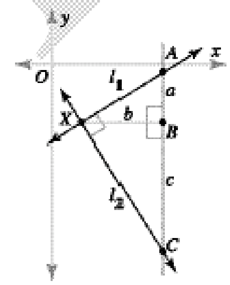
Note: Suppose that two lines l_1 and l_2 are parallel (as in the diagram). Then by the corresponding angles property, the angles between these lines and the x-axis are equal. Since each is a right triangle, it follows from the AA criterion that the two triangles are similar. Therefore, $h_1/h_2 = b_1/b_2$ and consequently, $h_1/b_1 = h_2/b_2$. But the slope of $l_1 = h_1/b_1$ and the slope of $l_2 = h_2/b_2$. Therefore the slope of l_1 equals the slope of l_2.



To demonstrate the converse, suppose lines l_1 and l_2 have the same slope. Then $h_1/b_1 = h_2/b_2$, so the triangles are similar by the SAS criterion. It follows that the angles between the lines and the x-axis are equal. By the corresponding angles property, the lines must be parallel.

- Two lines in a coordinate plane are perpendicular if and only if the product of their slopes is -1.

Note: Suppose that lines l_1 and l_2 are perpendicular and that he slope of l_1 is a/b and the slope of l_2 is $-c/b$. Triangles XAB and CAX are right triangles and share angle A. Therefore $\angle AXB = \angle XCB$, so triangle XAB is similar to triangle XBC by the AA criterion for similarity. It follows that $a/b = b/c$, and therefore $(a/b) \times (-c/b) = -1$.



7.8c Find the slopes of physical objects (roads, roofs, ramps, stairs) and express the answers as a decimal, ratio, or percent.

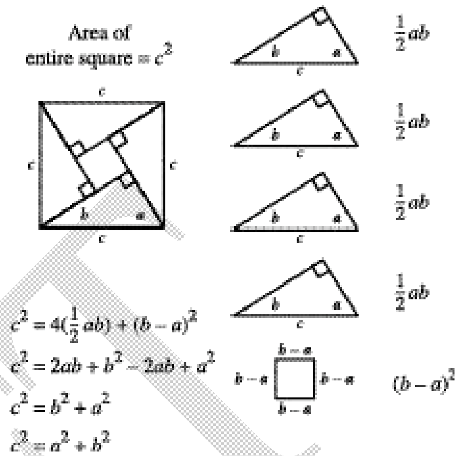
Geometry

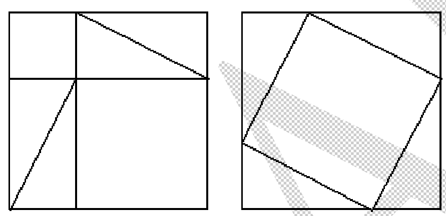
8.4 Understand and use the Pythagorean theorem .

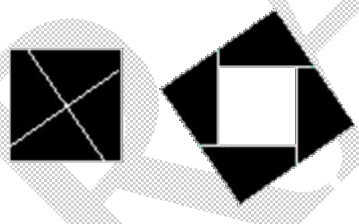
8.4a Prove the Pythagorean Theorem and its converse.

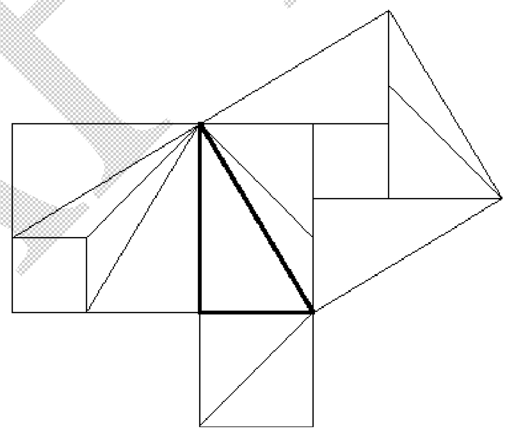
- A suggested proof of the Pythagorean Theorem using simple algebra is shown on the right.
- Other proofs using geometric dissection are shown below.

Note: The validity of these proofs depends on the fundamental fact that the sum of the angles in a triangle is 180°.









8.4b Use the Pythagorean theorem and its converse to find distances between points in the Cartesian coordinate system to solve perimeter, area, and volume problems.

- To make use of the Pythagorean theorem, it is often helpful to draw new lines in a figure in order to create right triangles.

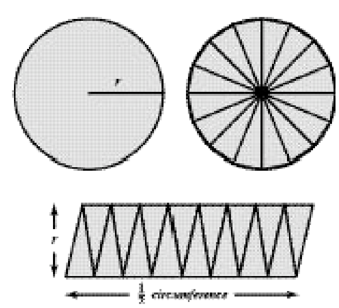
  Example: Use the fact that the bisector of the angle between the equal sides of an isosceles triangle is the perpendicular bisector of the opposite side.

8.5 Understand fundamental properties of a circle.

8.5a Understand and explain relationships among the radius, diameter, circumference, and area of a circle.

- The circumference of a circle is directly proportional to its radius, and to its diameter.

  Note: The circumference C of a circle can be thought of intuitively as the limit as the number of sides increases of the perimeters of an inscribed regular polygon. These perimeters are formed from



the bases of isosceles triangles whose two equal sides are radii of the circle. By similarity, if the radius of a circle increases, the bases of these triangles will increase in direct proportion.

- The area of a circle equals its radius times one-half of its circumference.

    Note: The area A of a circle can be thought of intuitively as the limit as the number of sides increases of the area of an inscribed regular polygon. The triangular pieces that make up one of these many-sided polygons can be rearranged alternately to form a rectangular strip whose height is approximately the radius r and whose base is approximately half the circumference C. Thus $A = r(C/2)$.

- The area of a circle is directly proportional the square of its radius (and also to the square of its diameter).

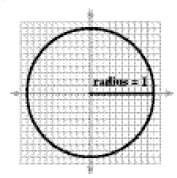
    Note: This follows directly from the two preceding statements. $C = kr$; $A = rC/2$; thus $A = kr^2/2$.

8.5b The ratio of the circumference to the diameter of a circle is the same as the ratio of the area to the square of the radius. This ratio is called pi, or π.

Note: Writing $C = kd = 2kr$, we see that $A = r(C/2) = kr^2$. Thus $C:d = A:r^2$.

- It follows that the area of a unit circle (one whose radius is 1) is π.

    Note: Thus one way to approximate π is to use graph paper to approximate the area of a unit circle.



- The value of π is approximately 3.14, or 22/7, or $3\frac{1}{7}$.
- Recognize the π is irrational.

8.5c Recognize that a triangle inscribed on the diameter of a circle is a right triangle.

8.5d Recognize that a tangent to a circle forms a right angle with the diameter at the point of tangency.

8.5e Know and use formulas for the circumference and area of a circle, semicircle, and quarter-circle.

8.5g Understand the definition of a great circle on a sphere.

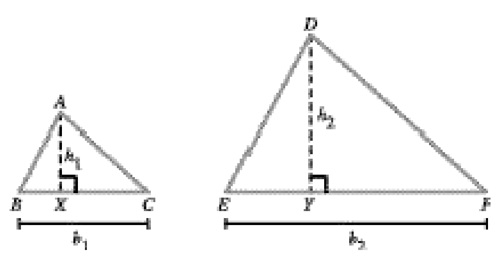
- Recognize that great circles provide shortest routes between points on the surface of a globe (or the earth).

8.6 Understand properties of scaling, dilation, and their relation to similarity.

8.6a Understand that similar polygons with scale factor r have areas related by a factor of r^2.

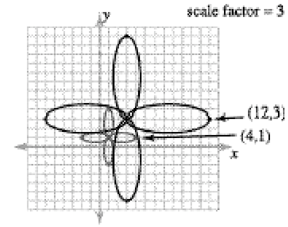
- Saying that two triangles are similar with *scale factor r* means that the ratio of corresponding sides is r.

    Note: In triangles similar triangles ABC and *DEF*, the ratios of bases b_1/b_2 and of heights h_2/h_1 equal the scale factor r. (Indeed, the ratios of all corresponding linear dimensions equal the scale factor r.) Thus the ratio of the areas of triangles DEF and ABC equals $b_2h_2/b_1h_1 = r^2$.



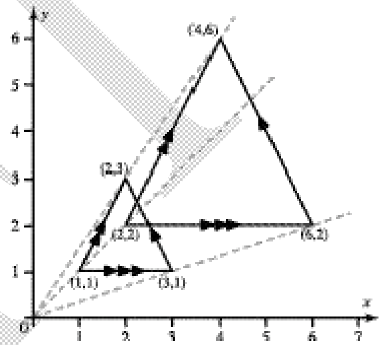
- Since any polygon can be decomposed into a finite number of triangles, it follows that any two similar polygons with scale factor r have areas related by a factor of r^2.
- By extension, volume expands as the cube of the scale factor.

8.6b Understand the definition and properties of dilation, and use it to define similarity of general figures in the plane.



- Informally, dilations are projections of figures in the plane from the perspective of the origin of a coordinate system.
- By definition, a *dilation* centered at the origin with scale factor *r* maps the point (x, y) to the point (rx, ry.)
- A dilation maps a line to a line with the same slope

  Note: The slope of a line determined by points (x_1, y_1) and (x_2, y_2) equals $(y_2 - y_1)/(x_2 - x_1)$. The slope of a line through the points (rx_1, ry_1) and (rx_2, ry_2) created by a dilation of scale factor *r* equals $(ry_2 - ry_1)/(rx_2 - rx_1) = (y_2 - y_1)/(x_2 - x_1)$.



- Dilations map parallel lines to parallel lines (except for those passing through the origin, which do not change).

  Note: Lines with the same slope are parallel (or concurrent).
- A dilation maps a triangle to a similar triangle.
- If two triangles in a coordinate system are similar, then there is a dilation moving one of them to a triangle congruent to the other.
- The concept of dilation can be used to define the similarity of arbitrary plane figures.

  Note: Previously, similarity was defined in terms of angles and ratios of line segments, limiting its applicability to polygonal shapes. Using dilation, similarity can be thought of as "projection from a point." This defines similarity for planar figures of any shape. (For computational simplicity, we use the origin as the projection point, but any other point would do.)

8.7 Solve problems involving perimeter, area, volume and surface area.

8.7a Find the perimeter or area of regions in the plane bounded by line segments and circular arcs.

- Use decomposition and triangulation to break a problem into manageable pieces, and use a calculator as necessary for calculations.

  Note: To maintain both accuracy and understanding, maintain exact formal calculations until the last step when a calculator should used to determine (or check) a decimal answer, which in many cases may be just an approximation.

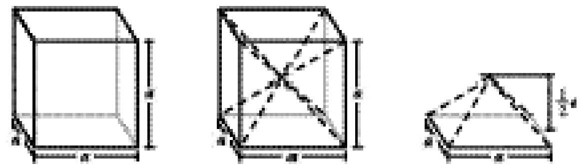
8.7b Understand relationships among volumes of common solids.

- Recognize the 3:2:1 relationship between the volumes of circular cylinders, hemispheres, and cones of the same height and circular base.

  Note: These relationships can be effectively demonstrated by pouring water or sand into properly sized plastic molds. They can also be calculated from the formulas $V = Ah$, $V = (2/3)Ah$, and $V = (1/3)Ah$ for the volumes of a right circular cylinder, hemisphere, and cone of height h and base area A.

- Recognize that the volume of a pyramid is one-sixth of the volume of the solid that can be decomposed into six identical pyramids.

  Note: Consider the special case of cube of side *a* that has been decomposed into 6 congruent pyramids. The height of each pyramid is $a/2$, the area of the base of each pyramid is a^2, and the volume of each pyramid is one-sixth the volum of the cube. Hence the volume is $a^3/6 = (1/3)\, a^2 \bullet (a/2) = (1/3)Ah$ where A is the area of the base and h is the height.

- Solve problems involving volumes and surface areas of common solids, including upright cylinders, spheres, cones, pyramids, and rectangular prisms.