Achieve, Inc.

MAP Mathematics Expectations

Number & Operations

K.1 Count objects and use numbers to express quantity.

K.1a Count up to 25 objects and tell how many there are in the counted group of objects.

Note: Accuracy depends on not skipping objects or counting objects twice. Counting objects foreshadows the important mathematical concept of *one-to-one correspondence*.

Note: Going past 20 is important to move beyond the irregular "teen" pattern into the regular twenty-one, twenty-two, ... counting routine.

K.1b Read aloud numerals 1 through 25 and match numerals with the numbers used in counting.

Note: In early grades "number" generally means "natural number" or, more mathematically, "non-negative integer."

K.1c Place numbers 1 through 25 in their correct sequence.

Note: The emphasis in kindergarten is on the sequence of numbers as discrete objects. The "number line" that displays continuous connection from one number to the next is introduced in Grade 2.

K.1d Count to 20 by twos.

- Recognize 20 as two groups of ten and as ten groups of two.

K.1e Recognize and use ordinal numbers (e.g., first, fourth, last).

Example: The fourth lady bug is about to fly.

K.2 Use number notation and place value up to 20.

K.2a Understand that numbers 1 through 9 represent "ones."

K.2b Understand that numbers 11 through 19 consist of one "ten" and some "ones."

- Relate the "teen" number words to groups of objects ("ten" + some "ones").

Example: 13 can be called "one ten and three ones," with "thirteen" being a kind of nickname.

K.3 Compare numbers up to 10

K.3a Compare sets of ten or fewer objects and identify which are equal to, more than, or less than others.

- Compare by matching and by counting.
- Use picture graphs (pictographs) to illustrate quantities being compared.

K.3b Recognize zero (0) as the count of "no objects."

Note: Zero is the answer to "how many are left?" when all of a collection of objects have been taken away.

Example: Zero is the number of buttons left after 7 buttons are removed from a box that contains 7 buttons.

K.4 Understand addition as putting together and subtraction as breaking apart.

K.4a Add and subtract single digit numbers whose total or difference is between 0 and 10.

- Write expressions such as 5 + 2 or 7 - 3 to represent situations involving sums or differences of numbers less than 10.

K.4b Understand add as "put together" or "add onto" and solve addition problems with numbers less than 10 whose totals are less than 20.

- Understand the meaning of addition problems phrased in different ways to reflect how people actually speak.
- Use fingers and objects to add.
- Attach correct names to objects being added.

  Note: This is especially important when the objects are dissimilar. For example, the sum of 3 apples and 4 oranges is 7 fruits.

K.4c Understand subtract as "break apart" or "take away" and solve subtraction problems using numbers between 1 and 10.

- Understand the meaning of addition problems phrased in different ways to reflect how people actually speak.

  Example: 7 – 3 equals the number of buttons left after 3 buttons are removed from a box that contains 7 buttons.

- Recognize subtraction situations involving missing addends and comparison.
- Use fingers, objects, and addition facts to solve subtraction problems.

K.4d Express addition and subtraction of numbers between 1 and 10 in stories and drawings.

- Translate such stories and drawings into numerical expressions such as 7 + 2 or 10 – 8.
- Model, demonstrate (act out), and solve stories that illustrate addition and subtraction.

K.5 Compose and decompose numbers 2 through 10.

K.5a Understand that numbers greater than 2 can be decomposed in several different ways.

Note: Decomposition and composition of single digit numbers into other single-digit numbers is of fundamental importance to develop meaning for addition and subtraction.

Example: 5 = 4 + 1 = 3 + 2; 10 = 9 + 1 = 8 + 2 = 7 + 3 = 6 + 4 = 5 + 5.

- Recognize 6 through 10 as "five and some ones."

  Note: This is an important special case because of its relation to finger counting.

  Example: 6 = 5 + 1; 7 = 5 + 2; 8 = 5 + 3; 9 = 5 + 4; 10 = 5 + 5.

Number & Operations

1.1 Understand and use number notation and place value up to 100.

1.1a Count to 100 by ones and tens.

- Group objects by tens and ones, and relate written numerals to counts of the groups by ones, and to counts of the groups by tens.

1.1b Read and write numbers up to 100 in numerals.

- Understand and use numbers up to 100 expressed orally.
- Write numbers up to 10 in words.

1.1c Recognize the place value of numbers (tens, ones).

- Recognize the use of *digit* to refer to the numerals 0 through 9.
- Arrange objects into groups of tens and ones and match the number of groups to corresponding digits in the number that represents the total count of objects.

1.2 Compare numbers up to 100 and arrange them in numerical order.

1.2a Arrange numbers in increasing and decreasing order.

1.2b Locate numbers up to 100 on the discrete number line.

- Understand that on the number line bigger numbers appear to the right of smaller numbers.

  Note: The *discrete number line* is not the continuous number line that will be used extensively in later grades, but a visual device for holding numbers in their proper regularly spaced positions The focus in Grades K-2 is on the uniformly spaced natural numbers, not on the line that connects them. However, for simplicity, in these grades the discrete number line is often called the *number line*.

- Use the number line to create visual representations of sequences.

  Examples: Even numbers, tens, multiples of five.

- Understand and use relational words such as *equal, bigger, greater, greatest, smaller,* and *smallest*; and phrases *equal to, greater than, more than, less than,* and *fewer than*.

1.2c Compare two or more sets of objects in terms of differences in the number of elements.

- Use matching to establish a one-to-one correspondence, and count the reminder to determine the size of the difference.
- Connect the meanings of relational terms (bigger, etc.) to the order of numbers, to the measurement of quantities (length, volume, weight, time), and to the operations of adding and subtracting.

  Example: If you add something bigger, the result is bigger, but if you take away something bigger, the result will be smaller.

1.3 Add, subtract, compose, and decompose numbers up to 100.

1.3a Be able to solve problems that require addition and subtraction of numbers up to 100 in a variety of ways.

- Know addition and subtraction facts for numbers up to 12.
- Add and subtract efficiently, both mentally and with pencil and paper.

Note: Avoid sums or differences that require numbers greater than 100 or less than 0.

- Be able to explain why the method used produces the correct answer.

  Note: Any correct method will suffice; there is no reason to insist on a particular algorithm since there are many correct methods. Common methods include "adding on" (often using fingers) and regrouping to make a ten.

  Examples: 6 + 8 = 6 + 4+ 4 = 10 + 4 = 14; or
  6 + 8 = 8 + 6 = 8 + 2 + 4 = 10 + 4 = 14;
- Add three single-digit numbers.

  Examples: 3 + 4 + 1 = ?; 7 + 5 + 3 = ?
- Understand and solve oral problems with a variety of phrasing, including *how many more*, or *how many fewer.*
- Know how to use a calculator to check answers.

1.3b Understand how to compose and decompose numbers.

- Identify and discuss patterns arising from decompositions.

  Example: *8 = 7 + 1 = 1+ 7 = 6 + 2 = 2+ 6 = 5 + 3 = 3+ 5 = 4 + 4;*
  9 = 8 + 1 = 1+ 8 = 7 + 2= 2 + 7= 3 + 6 = 6 + 3 = 5 + 4 = 4 + 5.
- Represent decomposition situations using terms such as *put together, add to, take from, break apart,* or *compare.*

1.3c Use groups of tens and ones to add numbers greater than ten.

- Using objects or drawings, add the tens, add the ones, and regroup if needed.

  Note: Grouping relies on the commutative and associative properties of addition. Examples in early grades foreshadow more formal treatments later. The vocabulary should await later grades.

  Examples:
  (a) 17 + 24 = 17 + 23 + 1 = 17 + 3 + 20 +1 = 20 + 20 + 1 = 40 + 1 = 41.
  (b) 58 + 40 = 50 + 8 + 40 = 50 + 40 + 8 = 98
  (c) 58 + 6 = 50 + 8 + 6 = 50 + 14 = 50 + 10 + 4 = 60 + 4 = 64
  (d) 58 + 26 = (58 + 2) + (26 - 2) = 60 + 24 = 84.

1.3d Create and solve addition and subtraction problems with numbers smaller than 20.

- Create and discuss problems using drawings, stories, picture graphs, diagrams, symbols, and open equations (e.g., 4 + ? = 17).
- Use the (discrete) number line to illustrate the meaning of addition and subtraction.
- Express answers in a form (verbal or numerical) that is appropriate to the original problem.
- Always check that answers are intuitively reasonable.

Number & Operations

2.1 Understand and use number notation and place value up to 1000.

2.1a Count by ones, twos, fives, tens, and hundreds.

- Count accurately for at least 25 terms.

  Example: Count by tens from 10 to 200; count by 2s from 2 to 50.

- Begin counts with numbers other than 1.

  Example: Count by tens from 200 to 300; count by 5s from 50 to 100.

2.1b Read and write numbers up to 1,000 in numerals and in words.

- Up to 1,000, read and write numerals, understand and speak words; write words up to 100.

2.1c Recognize the place values of numbers (hundreds, tens, ones).

- Understand the role of zero in place value notation.

  Example: In 508 = 5 hundreds, 0 tens, and 8 ones, the 0 tens cannot be ignored (even though it is equal to zero), because in place value notation it is needed to separate the hundreds position from the ones position.

  Note: Grade 2 begins the process of numerical abstraction--of dealing with numbers beyond concrete experience. Place value, invented in ancient India, provides an efficient notation that makes this abstract process possible and comprehensible.

2.1d Understand and utilize the relative values of the different number places.

- Recognize that the hundreds place represents numbers that are ten times as large as those in the tens place, and that the units place represents numbers that are ten times smaller than those in the tens place.

  Note: Understanding these relative values provides the foundation for understanding rounding, estimation, accuracy, and significant digits.

- Use meter sticks and related metric objects to understand how the metric system mimics the "power of ten" scaling pattern that is inherent in the place value system.

  Example: Write lengths, as appropriate, in centimeters, decimeters, meters, and kilometers.

2.1e Compare numbers up to 1,000.

2.2 Locate and interpret numbers on the number line.

2.2a Recognize the continuous interpretation of the number line where points correspond to distances from the origin (zero).

- Know how to locate zero on the number line.

  Note: The number line is an important unifying idea in mathematics. It ties together several aspects of number, including size, distance, order, positive, negative, and zero. Later it will serve as the basis for understanding rational and irrational numbers, and after that for the limit processes of calculus. In Grade 2 the interpretation of the number line advances from discrete natural numbers to a continuous line of indefinite length in both directions. Depending on context, a number N (e.g., 1 or 5) can be thought of either as a single point on the number line, or as the interval connecting the point 0 to the point N, or as the length of that interval.

2.2b Use number line pictures and manipulatives to illustrate addition and subtraction as the adding and subtracting of lengths.

Note: A meter stick marked in centimeters is a useful model of the number line because it reflects the place value structure of the decimal number system.

2.2c Understand the symbol $^1/_2$ and the word *half* as signifying lengths and positions on the number line that are midway between two whole numbers.

- Read foot and inch rulers with uneven hash marks to the nearest half inch.

2.3 Add, subtract, and use numbers up to 1000.

2.3a Add and subtract two- and three-digit numbers with efficiency and understanding.

- Add and subtract mentally with ones, tens, and hundreds.
- Use different ways to regroup or ungroup (decompose) to efficiently carry out addition or subtraction both mentally and with pencil and paper.

  ***Example**:* 389 + 492 = (389-8) + (8 + 492) = 381 + 500 = 881
- Perform calculations in writing and be able to explain reasoning to classmates and teachers.
- Add three two-digit numbers in a single calculation.
- Before calculating, estimate answers based on the left-most digits; after calculating use a calculator to check the answer.

2.3b Understand "related facts" associated with adding and subtracting.

Note: The expression "related facts" refers to all variations of addition and subtraction facts associated with a particular example.

- Solve addition equations with unknowns in various positions.

  Example: 348 + 486 = ?, 348 + ? = 834, ? + 486 = 834, 834 – 486 = ?
  834 – ? = 348, and ? – 486 = 348.
- Demonstrate how carrying (in addition) and borrowing (in subtraction) relate to composing and decomposing (or grouping and ungrouping)
- Connect the rollover cases of carrying in addition to the remote borrowing cases in subtraction.

  Example: 309 + 296 = 605; 605-296 = 309.

2.3c Create stories, make drawings, and solve problems that illustrate addition and subtraction with unknowns of various types.

- Understand situations described by phrases such as *put together* or *add to (*for addition) *take from, break apart,* or *compare* (for subtraction).
- Recognize and create problems using a variety of settings and language.

  ***Caution*:** Avoid being misled by (or dependent on) stock phrases such as *more* or *less* as signals for adding or subtracting.

2.3d Solve problems that require more than one step and that use numbers below 50.

Note. Since the challenge here is to deal with multi-step problems, the numbers are limited to those already mastered in the previous grade.

- Solve problems that include irrelevant information and recognize when problems do not include sufficient information to be solved.
- Represent problems using appropriate graphical and symbolic expressions.
- Express answers in verbal, graphical, or numerical form, using appropriate units.
- Check results by estimation for reasonableness, and by calculator for accuracy.

2.4 Understand multiplication as repeated addition, and division as the inverse of multiplication.

2.4a Multiply small whole numbers by repeated addition.

- Skip count by steps of 2, 3, 4, 5, and 10 and relate patterns in these counts to multiplication.

  Example: 3 × 4 is the 3rd number in the sequence 4, 8, 12, 16, 20, ...

- Relate multiplication by 10 to the place value system.

2.4b Understand division as the inverse of multiplication.

- Use objects to represent division of small numbers.

  Note: As multiplication is repeated addition, so division is repeated subtraction. Consequently, division reverses the results of multiplication and *vice versa*.

  Note: Since division is defined here as the inverse of multiplication, only certain division problems make sense, namely those that arise from a multiplication problem.

  Example: 8 ÷ 4 is 2 since 4 × 2 = 8, but 8 ÷ 3 is not (yet) defined.

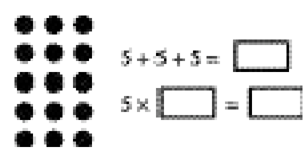
2.4c Know the multiplication table up to 5 × 5.

- Use multiplication facts within the 5 × 5 table to solve related division problems.

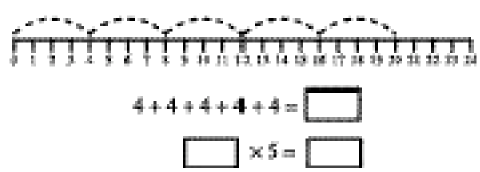
  Note: Multiplication facts up to 5 x 5 are easy to visualize in terms of objects or pictures, so introducing it in Grade 2 lays the foundation for the more complex 10 x 10 multiplication expectation that is central to Grade 3.

2.4d Solve multiplication and division problems involving repeated groups and arrays of small whole numbers.

- Arrange groups of objects into rectangular arrays to illustrate repeated addition and subtraction.
- Rearrange arrays to illustrate that multiplication is commutative.



- Demonstrate skip counting on the number line and then relate this representation of repeated addition to multiplication.



Number & Operations

3.1 Read, write, add, subtract, and comprehend five-digit numbers.

3.1a Read and write numbers up to 10,000 in numerals and in words.

3.1b Understand that digits in numbers represent different values depending on their location (place) in the number.

- Identify the thousands, hundreds, tens, ones positions and state what quantity each digit represents.

  Example: 9725 – 9325 = 400 because 7 - 3 = 4 in the hundreds position.

3.1c Compare numbers up to 10,000.

- Understand and use the symbols $<$, $\leq$, $>$, $\geq$ to signify order and comparison.
- Note especially the distinction between $<$ and $\leq$, and between $>$, $\geq$.

  Example: There are 6 numbers that could satisfy $97 < ? \leq 103$, but only five that could satisfy $97 < ? < 103$.

3.1d Understand and use grouping for addition and ungrouping for subtraction.

- Recognize and use the terms *sum* and *difference*.
- Use parentheses to signify grouping and ungrouping.

  Example: $375 + 726 = (3+7) \times 100 + (7+2) \times 10 + (5+6)$

  $= 10\times100 + 9\times10 + 10+1 = 10\times100 + 10\times10 + 1$

  $=(1 \times 1000) + (1 \times 100) + 0 \times 10 + 1 = 1101$

3.1e Add and subtract two-digit numbers mentally.

- Use a variety of methods appropriate to the problem, including adding or subtracting the smaller number by mental (or finger counting); regrouping to create tens; adding or subtracting an easier number and then compensating; creating mental pictures of manual calculation, and others.
- Check answers with a different mental method and compare the efficiency of different methods in relation to different types of problems.

3.1f Judge the reasonableness of answers by estimation.

- Use highest order place value (e.g., tens or hundreds digit) to make simple estimates.

3.1g Solve a variety of addition and subtraction problems.

- Story problems posed both orally and in writing.
- Problems requiring two or three separate calculations.
- Problems that include irrelevant information.

3.2 Multiply and divide with numbers up to 10.

3.2a Understand division as an alternative way of expressing multiplication.

- Recognize and use the terms *product* and *quotient*.
- Express a multiplication statement in terms of division, and vice versa.

  Example: $3 \times 8 = 24$ means that $24 \div 3 = 8$, and that $24 \div 8 = 3$.

3.2b Recognize different interpretations of multiplication and division and explain why they are equivalent.

- Understand multiplication as repeated addition, as area, and as the number of objects in a rectangular array.

> ***Example:*** Compare a class with 4 rows of 9 seats, a sheet of paper that is 4 inches wide and 9 inches high, and a picnic with 4 groups of 9 children each. Contrast with a class that has 9 rows of 4 seats, a sheet of paper that is 9 inches wide and 4 inches high, and a picnic that involves 9 groups of 4 children each.

- Understand division as repeated subtraction that inverts or "undoes" multiplication.
- Understand division as representing the number of rows or columns in a rectangular array, as the number of groups resulting when a collection is partitioned into equal groups, and as the size of each such group.

> ***Example*:** When 12 objects are partitioned into equal groups, 3 can represent either the number of groups (because 12 objects can be divided into three groups of four [4, 4, 4]) or the size of each group (because 12 objects can be divided into four groups of three [3, 3, 3, 3]).
>
> ***Note:*** In early grades use only ÷ as the symbol for division—to avoid confusion when the slash (/) is introduced as the symbol for fractions.

3.2c Know the multiplication table up to 10 × 10.

- Knowing the multiplication table means being able to find quickly missing values in open multiplication or division statements such as 56 ÷ 8 = [ ], 7 × [ ] = 42, or 12÷[ ] = 4.

> ***Note:*** Knowing by instant recall is the goal, but recalling patterns that enable a correct rapid response is an important early stage in achieving this skill.

3.3d Count aloud the first 10 multiples of each 1-digit natural number.

3.2e Create, analyze, and solve multiplication and division problems that involve groups and arrays.

- Describe contexts for multiplication and division facts.
- Complete sequences of multiples found in the rows and columns of multiplication tables up to 15 by 15.

3.2f Make comparisons that involve multiplication or division.

3.3 Solve contextual, experiential, and verbal problems that require several steps and more than one arithmetic operation.

> ***Note*:** Although solving problems is implicit in every expectation (and thus often not stated explicitly), this particular standard emphasizes the important skill of employing two different arithmetical operations in a single problem.

3.3a Represent problems mathematically using diagrams, numbers, and symbolic expressions.

3.3b Express answers clearly in verbal, numerical, or graphical (bar or picture) form, using units whenever appropriate.

3.3c Use estimation to check answers for reasonableness, and calculators to check for accuracy.

> ***Note*:** Problem selection should be guided by two principles: To avoid excess reliance on verbal skills, use real contexts as prompts as much as possible. And to focus on problem-solving skills, keep numbers simple, typically within the computational expectations one grade earlier.

3.4 Recognize negative numbers and fractions as numbers and know where they lie on the number line.

3.4a Know that symbols such as −1, −2, −3 represent *negative numbers* and know where the fall on the number line.

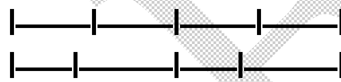
- Recognize negative numbers as part of the scale of temperature.
- Use negative numbers to count backwards below zero.
- Observe the mirror symmetry in relation to zero of positive and negative numbers.

  Caution: In grade 3, negative numbers are introduced only as names for points to the left of zero on the number line. They are not used in arithmetic at this point (e.g., for subtraction). In particular the minus sign (-) prefix on negative numbers should not at this stage be interpreted as subtraction.

3.4b Understand that symbols such as $^1/_2$, $^1/_3$, and $^1/_4$ represent numbers called *unit fractions* that serve as building blocks for all fractions.

- Understand that a unit fraction represents the length of a segment that results when the *unit interval* from 0 to 1 is divided into pieces of equal length.

  Note: A unit fraction is determined not just by the number of parts into which the unit interval is divided, but by the number of equal parts. For example, in the upper diagram that follows, each of the four line segments represents $^1/_4$, but in the lower diagram none represents $^1/_4$.

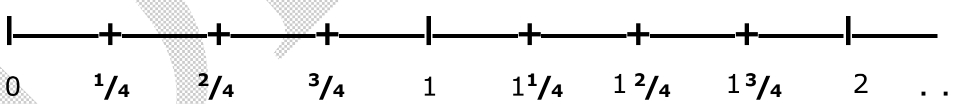


- Recognize, name, and compare unit fractions with denominators up to 10.

  Example: The unit fraction $^1/_6$ is smaller than the unit fraction $^1/_4$ since when the unit interval is divided into 6 equal parts, each part is smaller than if it were divided into four equal parts. The same thing is true of cookies or pizzas: one-sixth of something is smaller than one-fourth of that same thing.

3.4c Understand that each unit fraction $^1/_n$ generates other fractions of the form $^2/_n$, $^3/_n$, $^4/_n$, ... and know how to locate these fractions on the number line.

- Understand that $^1/_n$ is the point to the right of 0 that demarcates the first segment created when the unit interval is divided into n equal segments. Points marking the endpoints of the other segments are labeled in succession with the numbers $^2/_n$, $^3/_n$, $^4/_n$, ... These points represent the numbers that are called *fractions*.



- Understand that a fractional number such as $^1/_3$ can be interpreted either as the point that lies one-third of the way from 0 to 1 on the number line or as the length of the interval between 0 and this point.

  Note: When the unit interval is divided into n segments, the point to the right of the last (n^{th}) segment is n/n. This point, the right end-point of the unit interval, is also the number 1.

3.5 Understand, interpret, and represent fractions.

3.5a Recognize and utilize different interpretations of fractions, namely, as a *point* on the number line; as a *number* that lies between two consecutive (whole) numbers; as the *length* of a segment of the real number line; and as a *part* of a whole.

***Note*:** The standard of meeting this expectation is not that children be able to explain these interpretations but that they are able to use different interpretations appropriately and effectively.

3.5b Understand how a general fraction *n*/*d* is built up from *n* unit fractions of the form 1/d.

- Understand and use the terms *numerator* and *denominator.*
- Understand that the fraction *n/d* is a number representing the total length of *n* segments created when the unit interval from 0 to 1 is divided into *d* equal parts.

  Note: This definition applies even when *n* > *d* (i.e., the numerator is greater than the denominator): just lay *n* segments of size *d* end to end. It will produce a segment of length *n/d* regardless of whether *n* is less than, equal to, or greater than *d*. Consequently, there is no need to require that the numerator be smaller than the denominator.
- Recognize that when *n* = *d,* the fraction *n*/*d* =1; when *n* < *d*, *n*/*d* < 1; and when *n* > *d, n*/*d* > 1.

  Examples: *2/2 = 1, 2/3 < 1, and 3/2 > 1.*
- Recognize the associated vocabulary of *mixed number*, *proper fraction*, and *improper fraction*.

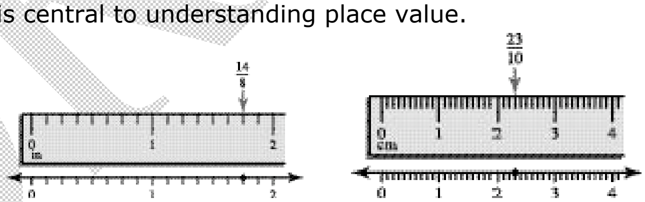
  Note: These terms are somewhat archaic and not of great significance. It makes no difference if the numerator of a fraction is larger than the denominator, so there is nothing "improper" about so-called "improper fractions."

3.5c Locate fractions with denominator 2, 4, 8, and 10 on the number line.

- Understand how to interpret mixed numbers with halves and quarters (e.g., $3^1/_2$ or $1^1/_4$) and know how to place them on the number line.

  Note: Measurement to the nearest half or quarter inch provides a concrete model.
- Use number lines and rulers to relate fractions to whole numbers.

  Note: The denominators 2, 4, and 8 appear on inch rulers and are created by repeatedly folding strips of paper; the denominator 10 appears on centimeter rulers and is central to understanding place value.



3.5d Understand and use the language of fractions in different contexts.

- When used alone, a fraction such as $^1/_2$ is a number or a length, but when used in contexts such as "$^1/_2$ of an apple" the fraction represents a part of a whole.

  Note: A similar distinction also applies to whole numbers: The phrase "I'll take 3 oranges" is not about taking the number 3, but about counting 3 oranges. Similarly, "1/2 of an orange" is not about the number (or unit fraction) 1/2, but is a reference to a part of the whole orange.

  Note: The vocalization of unit fractions (*one-half, one-third, one-fourth*) are expressions children will know from prior experience (e.g., one-half cup of sugar, one-quarter of an hour). Mathematical fractions extend this prior knowledge to numbers by dividing an interval of length 1. In this way, the unit fraction 1/2 can be defined as the number representing one-half of the unit interval.

3.5e Recognize fractions as numbers that solve division problems.

- When the unit interval is divided into equal parts to create unit fractions, the sum of all the parts adds up to the whole interval, or 1. In other words, the total of n copies of the unit fraction 1/n equals 1. Since division is defined as the inverse of multiplication, this is the equivalent of saying that 1 divided by n equals 1/n.

  Example: Since 4 copies of the unit fraction 1/4 combine to make up the unit interval, 4 × (1/4) = 1. Equivalently, 1 ÷ 4 = 1/4.

  ***Caution*:** At first glance, the statement "1 ÷ 4 = 1/4" might appear to be a tautology. It is anything but. Indeed, understanding why this innocuous equation is expressing something of importance is an important step in understanding fractions. The fraction 1/4 is the name of a point on the number line, the length of part of the unit interval. The open equation 1 ÷ 4 = ? asks for a number with the property that 4 × ? = 1. By observing that the four parts of the unit interval add up to the whole interval, whose length is 1, we discover that the length of one of these parts is the unknown needed to satisfy the equation: 4 x 1/4 = 1. this justifies the assertion that 1 ÷ 4 = 1/4.

3.6 Understand how to add, subtract, and compare fractions with equal denominators.

3.6a Recognize how adding and subtracting fractions with equal denominators can be thought of as the joining and taking away, respectively, of contiguous segments on the number line.

Note: Common synonyms for *equal* denominators are *common* denominators or *like* denominators or *same* denominators. The latter appear to emphasize the form of the denominator (e.g., all 4s) whereas "equal" correctly focuses on what matters, namely, the *value* of denominator.

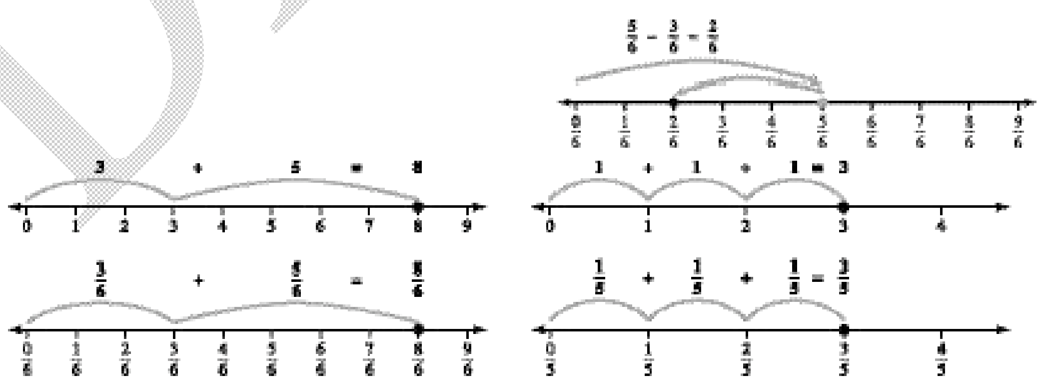
3.6b Understand that a fraction *n*/*d* is the sum of *n* unit fractions of the form 1/*d*.

Example: $\frac{3}{5} = \frac{1}{5} + \frac{1}{5} + \frac{1}{5}$.

3.6c Compare, add, and subtract fractions with equal denominators.

- Addition and subtraction of fractions with equal denominators work exactly as do addition and subtraction of whole numbers, and therefore builds on the addition and subtraction of whole numbers.

  Note: There is no need to simplify answers to lowest terms.



Number & Operations

4.1 Read, write, add and subtract positive whole numbers.

4.1a Read and write numbers in numerals and in words.

4.1b Recognize the place values in numbers and understand what quantities each digit represents.

- Understand that each digit represents a quantity ten times as great as the digit to its right.

4.1c Compare natural numbers expressed in place value notation.

4.1d Add columns consisting of several 3-4 digit numbers.

- Use and develop skills such as creating tens and adding columns first down then up to ensure accuracy.

  Example: The most common example is a list of prices (e.g., a grocery bill, or a shopping list).

- Check answers with a calculator.

4.2 Understand why and how to approximate or estimate.

4.2.a Round off numbers to the nearest 5, 10, 25, 100, or 1,000.

- *Rounding off* is something done to an overly exact number (e.g., a city's population given as 235,461). *Estimation* and *approximation* are actions taken instead of, or as a check on, an exact calculation. Estimates and approximations are almost always given as round numbers.

  Examples: In estimating the number of students to be served school lunch, round the number to the nearest 10 students. In estimating a town's population, rounding to the nearest 50 or 100 is generally more appropriate.

4.2.b Estimate answers to problems involving addition, subtraction, and multiplication.

4.2.c Judge the accuracy appropriate to given problems or situations.

- Use estimation to check the reasonableness of answers.
- Pay attention to the way answers will be used to determine how much accuracy is important.

  Note: There are no formal rules that work in all cases. This expectation is about judgment.

4.3 Identify small prime and composite numbers.

4.3a Understand and use the definitions of prime and composite number.

- Understand and use the terms *factor* and *divisor.*
- Apply these definitions to identify prime and composite numbers under 50.

  Note: A prime number is a natural number that has exactly two positive divisors, 1 and itself. A composite number is a natural number that has more than two divisors. By convention, 1 is neither prime nor composite.

4.3b List all factors of integers up to 50.

4.3c Determine if a 1-digit number is a factor of a given integer and whether a given integer is a multiple of a given 1-digit number.

- Find a common factor and a common multiple of 2 numbers.

***Note**:* Common factors and multiples provide a foundation for arithmetic of fractions and for the idea of greatest common factor and least common multiple which are developed in later grades.

4.3d Recognize that some integers can be expressed as a product of factors in more than one way.

Example: $12 = 4 \times 3 = 2 \times 6 = 2 \times 2 \times 3$.

4.4 Multiply small multi-digit numbers and divide by single digit numbers.

4.4a Understand and use a reliable algorithm for multiplying multi-digit numbers accurately and efficiently.

- Multiply any multi-digit number by a 1-digit number.
- Multiply a 3 digit number by a 2-digit number.
- Explain why the algorithm works.

Example: Justification of a multiplication algorithm relies on the distributive property applied to place value--an analysis that helps prepare students for algebra. For example, using the distributive property, 2×35 can be written as $2(30 + 5) = 60 + 10 = 70$. Here's how the analysis applies to a more complex problem: 258×35 can be written as $(200 + 50 + 8) \times 35$. This becomes:

$$200 \times 35 + 50 \times 35 + 8 \times 35 = 200(30 + 5) + 50(30 + 5) + 8(30 + 5).$$

From this point computations can be done mentally:

$$6000 + 1000 + 1500 + 250 + 240 + 40 = 9030.$$

4.4b Understand and use a reliable algorithm for dividing numbers by a single-digit number accurately and efficiently.

- Explain why the algorithm works.
- Understand division as fair shares and as successive subtraction, and explain how the division algorithm yields a result that conforms with these understandings.
- Check results both by multiplying and by using a calculator.

4.4c Recognize, understand, and correct common computational errors.

Examples: Common errors are displayed at the right.

```
  13 r 7
6|85
```

```
   26
 × 12
   52
 + 26
   78
```

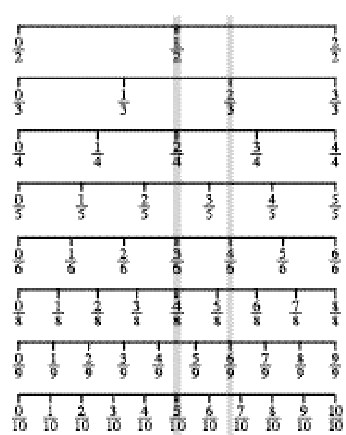
4.4d Understand the role and function of remainders in division.

- For whole numbers a, b, and c with $b \neq 0$,
  - -- when *a* is a multiple of *b,* the statement $a \div b = c$ is merely a different way of writing $a = c \times b$;
  - -- when a is not a multiple of b, the division $a \div b$ is expressed as $a = c \times b + r$, where the "remainder" r is a whole number less than b.

4.5 Understand and use the concept of equivalent fractions.

4.5a Understand that two fractions are equivalent if they represent the same number.

Note: Because equivalent fractions represent the same number, we often say, more simply, that they are the same, or equal.



***Examples**:* Just as 2 + 2 represents the same number as 4, so 4/6 represents the same number as 2/3. The diagram on the right shows that $\frac{1}{2}=\frac{2}{4}=\frac{3}{6}=\frac{4}{8}=\frac{5}{10}$ and $\frac{2}{3}=\frac{4}{6}=\frac{6}{9}$.

- Illustrate equivalent fractions using small numbers with both length and area.

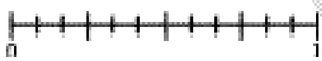
  Example: Figure 1 demonstrates in two different ways (length and area) how the fact that 3 x 3 = 9 and 3 x 4 = 12 makes 3/4 equivalent to 9/12.

Using length to illustrate equivalent fractions:

Let the whole be the length of a line segment. Divide it into 4 equal parts:

The length of each part represents 1/4 by the definition of the fraction 1/4. Therefore 3/4 is represented by the length of the thickened line segment, because it has 3 of the 4 equal parts.

Divide the length of each equal part of the whole into 3 equal parts:



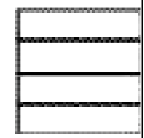
Here the length of each small line segment represents 1/12. Now 3/4 of the whole takes up 9 of these small line segments:

Therefore the thickened line segment represents 9/12. Since the thickened line segment also represents 3/4, we see that 3/4 equals

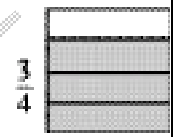
$$\frac{3\times 3}{3\times 4}=\frac{9}{12}.$$

Using area to illustrate equivalent fractions:

Let the whole be the area of a square. Divide it into 4 equal parts:

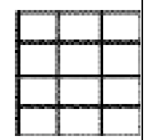


The area of each part represents 1/4 by the definition of the fraction 1/4. Therefore 3/4 is represented by the area of the shaded region:



Divide each equal part of the whole into 3 equal parts:

The area of each small rectangle represents 1/12. Now 3/4 of the area of the whole takes up 9 of these small rectangles:



Therefore the shaded area represents 9/12. Since the shaded area also represents 3/4, we see that 3/4 equals

$$\frac{3\times 3}{3\times 4}=\frac{9}{12}.$$

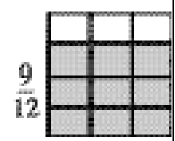


Figure 1

Note: Adults use three symbols interchangeably to represent division: ÷, /, and –. The latter two are also used interchangeably to represent fractions. Indeed, the symbol 2/3 is as often used to represent a fraction as the result of the act of division. In school, however, since fractions and division are introduced in a specific sequence, it is important that these not be used interchangeably until their equivalence has been well established and rehearsed.

4.5b Place fractions on the number line.

- Understand that equivalent fractions represent the same point on the number line.

  Note: As introduced in Grade 3, fractions can be interpreted as a *point* on the number line; as a *number* that lies between two consecutive whole numbers; as the *length* of a segment on the real number line; and as a *part* of a whole. Two fractions are *equivalent* in each of these interpretations if they refer to the same point, number, length, or part of a whole.

4.5c Understand that any two fractions can be written as equivalent fractions with equal denominators.

- Use length or area drawings to illustrate these equivalences.

Note: The phrase "like denominator" is often used in this context. However, it is equality, not form or "likeness," that is important.

Example: 1/3 and 5/15 are equivalent because both represent one-third of the unit interval. Similarly, 1/5 and 3/15 are also equivalent because both represent one-fifth of the unit interval.

Example: $\frac{5}{6}$ is equivalent to $\frac{5\times 7}{6\times 7}$, and $\frac{8}{7}$ is equivalent $\frac{8\times 6}{7\times 6}$, both of which have the same denominators.

Note: More generally, *a/b* and c/d are equivalent to the fractions $\frac{a\times d}{b\times d}$ and $\frac{c\times b}{d\times b}$ respectively. This shows a general method for transforming fractions into equivalent fractions with equal (common) denominators.

Note: The calculations that create equivalent fractions require multiplying both the numerator and the denominator separately, by the same number. This is, of course, the same as multiplying the fraction itself by 1--which is why the two fractions are equivalent, However, it is premature at this stage to suggest that students think of $\frac{5\times 7}{6\times 7}$ as $\frac{5}{6}\times 1$ because multiplication of fractions by whole numbers is not yet addressed.

4.5d Use equivalent fractions to compare fractions.

- Use the symbols < and > to make comparisons in both increasing and decreasing order.
- Emphasize fractions with denominators of 10 or less.

Example: The fractions 5/6 and 3/8 can be compared using the equivalent fractions $\frac{5\times 8}{6\times 8}$ and $\frac{3\times 6}{8\times 6}$.

4.6 Add and subtract simple fractions.

4.6a Add and subtract fractions by rewriting them as equivalent fractions with a common denominator.

- Solve addition and subtraction problems with fractions that are less than 1 and whose denominators are either (a) less than 10 or (b) multiples of 2 and 10, or (c) multiples of each other.
- Add and subtract lengths given as simple fractions (e.g., $^1/_3 + ^1/_2$ inches).
- Find the unknowns in equations such as: $^1/_8$ + [ ] = $^5/_8$ or $^3/_4$ - [ ] = $^1/_2$.

Note: The idea of common denominator is a natural extension of common multiples introduced above. Addition and subtraction of fractions with common denominators was introduced in Grade 3.

Note: To keep calculations simple, do not use mixed numbers (e.g., $3^1/_2$) or sums involving more than two different denominators (e.g., $^1/_3 + ^1/_2 + ^1/_5$). Also, do not stress reduction to a 'simplest' form (because, among many reasons, such forms may not be the simplest to use in subsequent calculations).

4.6b Recognize mixed numbers as an alternate notation for fractions greater than 1.

- Know how to interpret mixed numbers as an addition.
- Locate mixed numbers on the number line.

Example: $\frac{23}{4} = 5\frac{3}{4}$ because on the number line $\frac{23}{4}$ is $\frac{3}{4}$ to the right of 5.

4.7 Understand and use decimal numbers up to hundredths.

4.7a Understand decimal digits in the context of place value for terminating decimals with up to two decimal places.

- A *terminating decimal* is place value notation for a special class of fractions with powers of 10 in the denominators.
- Understand the values of the digits in a decimal and express them in alternative notations.

Examples: The terminating decimal 0.59 equals the fraction 59/100. Similarly, the decimal 12.3 is just another way of expressing the fraction 123/10 or the mixed number $12^3/_{10}$.

Note: Two-place decimals were introduced in Grade 3 to represent currency. The concept of two-place decimals as representing fractions with denominator 100 is equivalent to saying that the same amount of money can be expressed either as dollars ($1.34) or as cents (134¢).

Note: The denominators of fractions associated with decimal numbers, being powers of ten, are multiples of one another. This makes adding such fractions relatively easy. For example,

$$2.34 = \frac{234}{100} = \frac{200+30+4}{100} = \frac{200}{100} + \frac{30}{100} + \frac{4}{100} = \frac{2}{1} + \frac{3}{10} + \frac{4}{100}.$$

4.7b Add and subtract decimals with up to two decimal places.

- The arithmetic of decimals becomes arithmetic of whole numbers once they are rewritten as fractions with the same denominator:

$$0.5 + 0.12 = \frac{5}{10} + \frac{12}{100} = \frac{50}{100} + \frac{12}{100} = \frac{50+12}{100} = 62/100 = .62$$

- Add and subtract two-decimal numbers, notably currency values, in vertical form.

4.7c Write tenths and hundredths in decimal and fraction notation and recognize the fraction and decimal equivalents of halves, fourths and fifths.

Note: "Thirds" are missing from this list since 1/3 cannot be represented by a terminating decimal. This is because no power of 10 is a multiple of three, so the fraction 1/3 does not correspond to any terminating decimal.

4.7d Use decimal notation to convert between grams and kilograms, meters and kilometers, and cents and dollars.

4.8 Solve multi-step problems using whole numbers, fractions, decimals, and all four arithmetic operations.

4.8a Solve problems of various types (mathematical tasks, word problems, contextual questions, and "real-world" settings) that require more than one of the four arithmetic operations.

***Note**:* Problem-solving is an implied part of all expectations, but also sometimes worth special attention, as here where all four arithmetic operations are available for the first time. As noted earlier, to focus on strategies for solving problems that are cognitively more complex than those previously encountered, computational demands should be kept simple.

4.8b Understand and use parentheses to specify the order of operations.

- Know why parentheses are needed, when and how to use them, and how to evaluate expressions containing them.

4.8c Use the inverse relation between multiplication and division to check results when solving problems.

Example: Recognize that 185 ÷ 5 = 39 is wrong because 39 × 5 = 195.

- Use multiplication and addition to check the result of a division calculation that produces a non-zero remainder.

4.8d Translate a problem's verbal statements or contextual details into diagrams and numerical expressions, and express answers in appropriate verbal or numerical form, using units as needed.

4.8e Use estimation to judge the reasonableness of answers.

4.8f Create verbal and contextual problems representing a given number sentence and use the four operations to write number sentences for given situations.

Number & Operations

5.1 Understand that every natural number can be written as a product of prime numbers in only one way (apart from order).

5.1a Extend knowledge of prime and composite numbers up to 100.

- Write composite numbers up to 100 as a product of prime factors.

  Note: Prime and composite numbers were introduced in Grade 4. Here the goal is to investigate more examples to develop experience with larger numers.

5.1b Decompose composite numbers into products of factors in different ways and identify which of these combinations are products of prime factors.

- Recognize that every decomposition into prime factors involves the same factors apart from order.

  Note: It is this uniqueness ("the same factors apart from order") of the prime decomposition of integers that makes this fact important--so much so that this result is often called "the fundamental theorem of arithmetic."

  Examples: 24 = 2 × 12 = 2 x 3 x 4 = 2 x 3 x 2 x 2.
  24 = 3 x 8 = 3 x 4 x 2 = 3 x 2 x 2 x 2.
  24 = 4 x 6 = 2 x 2 x 2 x 3.

5.2 Know how to divide whole numbers.

5.2a Understand and use a reliable algorithm for division of whole numbers.

- Recognize that the division of a whole number *a* by a whole number *b* (symbolized as $a \div b$) is a process to find a *quotient q* and a *reminder r* satisfying $a = q \times b + r$, where both *q* and *r* are whole numbers and $r < b$.

  Note: The division algorithm most widely used in the United States is called *long division*. Although the term itself is often taken to mean division by a two digit number, the algorithm applies equally well to division of a multi-digit number by a single digit number.

- Understand that the long division algorithm is a repeated application of division-with-remainder.

  Example: To divide 85 by 6, write 85 = 80 + 5. Dividing 80 by 6 yields 6 10s with 20 left over. In other words, 85 = 80 + 5 = (10 × 6) + 20 + 5 = (10 × 6) + 25. In the long division algorithm, this is written as 6 in the tens place with a remainder of 25. Next, in long division, we divide the remainder 25 by 6: 25 = (4 × 6) + 1. Combining both steps yields 85 = (10 × 6) + 25 = (10 × 6) + (4 × 6) +1 = (14 × 6) + 1.

  $$\begin{array}{r} 14 \\ 6\overline{)85} \\ -60 \\ \hline 25 \\ -24 \\ \hline 1 \end{array}$$

  Note: Since long division is a process in which the same steps are repeated until an answer is obtained, the example just given offers sufficient understanding of the general process.

5.2b Divide numbers up to 1,000 by numbers up to 100 using long division or some comparable approach.

- Estimate accurately in the steps of the long division algorithm.

  Example: *To compute 6512 ÷ 27 requires knowing how many 27's there are in 65, in 111, and in 32.*

- Check results by verifying the division equation $a = q \times b + r$, both manually and with a calculator.

5.2c Know and use mental methods to calculate or estimate the answers to division problems.

- Mentally divide numbers by ten, one hundred, and one thousand.

- Where possible, break apart numbers before dividing to simplify mental calculations.

  Example: *Divide 49 by 4 by writing 49 = 48 + 1. Since 48/4= 12, 49 ÷ 4 = yields the quotient 12 and remainder 1*.

5.3 Understand how to add and subtract fractions.

5.3a Add fractions with unequal denominators by rewriting them as equivalent fractions with equal denominators.

Note: In Grade 4, addition of fractions was restricted to unit fractions, or to those in which one denominator was a multiple of the other. In both cases, these restrictions simplify the required calculations. Here the goal is to understand and learn to do the most general case.

- Understand and use the general formula $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$.

  Note: There is no need to find a *least* common denominator. The easiest common denominator of $\frac{a}{b}$ and $\frac{c}{d}$ is most often *bd*.

- When necessary, use calculators to carry out the required multiplications.

  Example: 17/19 + 13/14 = [(17 × 14) + (13 × 19)]/(19 × 14)
  = (238 + 247)/266 = 485/266

5.3b Add and subtract mixed numbers.

Example: $2\frac{1}{15}-1\frac{3}{4} = \frac{30}{15}+\frac{1}{15} - \left(\frac{4}{4}+\frac{3}{4}\right) = \frac{31}{15} - \frac{7}{4} = \frac{124}{60} - \frac{105}{60} = \frac{19}{60}$

5.3c Find the unknown in simple equations involving fractions and mixed numbers.

Examples: $2^2/_3$ + [ ] = $5^1/_4$; [ ] × 14 + 3 = 101.

5.4 Understand what it means to multiply fractions and know how to do it.

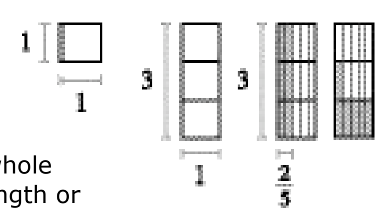
5.4a Understand how multiplying a fraction by a whole number can be interpreted as repeated addition of the fraction.

Example: 3 × $^2/_5$ can be thought of as $^2/_5$ + $^2/_5$ + $^2/_5$ = $^6/_5$.

Note: As introduced in Grade 3, fractions can be interpreted as a *point* on the number line; as a *number* that lies between two consecutive whole numbers; as the *length* of a segment on the real number line; and as a *part* of a whole. Defining multiplication of fractions by whole numbers as repeated addition is analogous to the how multiplication of whole numbers is understood, and readily conforms to the number and length interpretations of fractions.

- In general, if *a, b,* and *c* are whole numbers and *c* ≠ 0 then $a\times\frac{b}{c}=\frac{ab}{c}$.

  Note: In interpreting multiplication of a fraction by a whole number as repeated addition we introduce a curious asymmetry. 3 × $^2/_5$ is $^2/_5$ added to itself three times, but it does not make sense to think of $^2/_5$ × 3 as 3 being added to itself $^2/_5$ times. This leaves $^2/_5$ × 3 undefined under this interpretation. If we were sure that multiplication of fractions is commutative, as is multiplication of whole numbers, then we would be able to say that $^2/_5$ × 3 = 3 × $^2/_5$. But to do this requires the "part of a whole" interpretation of fractions.



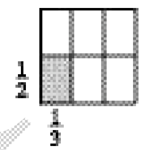
Example: The multiplication of a fraction by a whole number can also be interpreted by means of a length or area model. Here's an example of how using an area model for $3 \times {}^2/_5$. Taking the whole as the area of a unit square, 3×1 would be the area of the tower consisting of three unit squares. Since ${}^2/_5$ means dividing the whole into 5 equal parts and taking 2 of them, the sum ${}^2/_5 + {}^2/_5 + {}^2/_5$ can be represented by the shaded area in the middle figure on the right. The figure on the far right rearranges the small shaded rectangles to show that $3 \times {}^2/_5 = {}^2/_5 + {}^2/_5 + {}^2/_5 = {}^6/_5$.

5.4b Understand how multiplying two fractions can be interpreted in terms of an area model.

- Understand why the product of two unit fractions is a unit fraction whose denominator is the product of the denominators of the two unit fractions.

  Note: Taking the whole to be a unit square, then ${}^1/_a \times {}^1/_b$ is by definition the area of a rectangle with length ${}^1/_a$ and width ${}^1/_b$. In symbols, ${}^1/_a \times {}^1/_b = {}^1/_{ab}$.

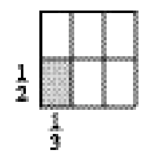
  Example: Let the whole be the area of a unit square. Then ${}^1/_2 \times {}^1/_3$ is by definition the area of a rectangle with sides of length ${}^1/_2$ and ${}^1/_3$. The shaded rectangle in this drawing of the unit square is such a rectangle. The shaded area is also ${}^1/_6$ of the area of the whole. Therefore, ${}^1/_2 \times {}^1/_3 = {}^1/_6$.



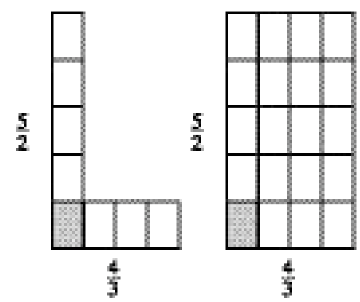
- Interpret the formula $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ in terms of area.

  Note: If the area of the whole is a unit square, then *a/b × c/d* is by definition the area of a rectangle with length *a/b* and width *c/d*.

  Example: To illustrate the multiplication ${}^5/_2 \times {}^4/_3$ using area models,



  let the whole be the area of a unit square. Then ${}^1/_2 \times {}^1/_3$ is the area of the shaded rectangle with length of side ${}^1/_2$ and width ${}^1/_3$. By definition, ${}^5/_2 = {}^1/_2 + {}^1/_2 + {}^1/_2 + {}^1/_2 + {}^1/_2$ and ${}^4/_3 = {}^1/_3 + {}^1/_3 + {}^1/_3 + {}^1/_3$. These are illustrated in the diagrams on the right. The large rectangle has been made from 5×4 copies of the small shaded rectangle shown above. Since ${}^1/_2$ and ${}^1/_3$ are unit fractions, the area of the shaded rectangle is ${}^1/_2 \times {}^1/_3 = {}^1/_{(2\times3)}$. Therefore the area of the large rectangle is $(5 \times 4)/(2 \times 3)$.



- Recognize that the formula $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ shows that the multiplication of fractions is commutative.

  Note: By validating commutativity of multiplication, the area model provides the crucial feature that is missing from the "repeated addition" model for multiplication of fractions. This shows that ${}^2/_5 \times 3 = 3 \times {}^2/_5$.

  Note: The formula for multiplying fractions can be used to show that fractions also obey the associative and distributive laws of whole number arithmetic. Experience with examples is sufficient to gain insight into just how this works.

5.4c Understand why "${}^a/_b$ of c" is the same as "${}^a/_b \times$ c."

Example: The phrase "${}^2/_5$ of 3" mean ${}^2/_5$ of a whole that is 3 units (e.g., ${}^2/_5$ of \$3, ${}^2/_5$ of 3 pizzas, ${}^2/_5$ of 3 cps of sugar). To take ${}^2/_5$ of 3 units, take ${}^2/_5$ of each unit and add them together: ${}^2/_5 + {}^2/_5 + {}^2/_5 = 3 \times {}^2/_5$. Since multiplication of fractions is commutative, $3 \times {}^2/_5 = {}^2/_5 \times 3$.

Example: 3/4 of the length of a 12-inch ruler is 9 inches, while 3/4 of the length of 100-centimeter ruler is 75 centimeters.

5.4d Understand that the product of a positive number with a positive fraction less than 1 is smaller than the original number.

- In symbols, if a, b, c, and d are all > 0 and $\frac{a}{b} < 1$ then $\frac{a}{b} \times \frac{c}{d} < \frac{c}{d}$.

  Note: Area is again the easiest model: *a/b* × *c/d* can be represented by a rectangle with dimensions *a/b* and *c/d*, whereas *c/d* can be represented by a rectangle of dimensions 1 and *c/d*. When *a/b < 1*, the former wll fit inside the latter, thus showing that it has smaller area.

5.5 Understand and use the interpretation of a fraction as division.

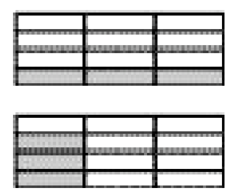
5.5a Understand why the fraction *a/b* can be considered an answer to the division problem *a* ÷ *b*.

- Among whole numbers, the answer to *a* ÷ *b* is a quotient and a remainder (which may be zero). Among fractions, the answer to *a* ÷ *b* is the fraction *a/b.*

  Note: The expression *a* ÷ *b* where *a* and *b* are whole numbers signifies a process to find a *quotient q* and a *reminder r* satisfying *a = q × b + r*, where both *q* and *r* are whole numbers and r < b. If we permit fractions as answers, then q = $^a/_b$ and r =0 will always solve the division problem since $a = {}^a/_b \times b + 0.$

  Note: This fact justifies using the fraction bar (— or /) to denote division rather than the division symbol (÷). Beyond elementary school, this is the common convention since the limitation of integer answers (quotient and remainder) is much less common.

  Example: To illustrate the assertion that a/b = *a* ÷ *b* with the fraction 3/4, begin as usual with the whole being a unit square. 3÷4 is the area of one part when three *wholes* are divided into 4 equal parts as shown. By moving all three shaded rectangles into the same whole, as shown, they form 3 parts of a whole that has been divided into 4 equal parts. That is the definition of the fraction 3/4. Thus 3÷4 = 3/4.

  Note: Another way to think about the relation between fractions and divisions is to begin with 4 × 3/4 = 3. This says that 4 equal parts, each of size 3/4, make up 3 wholes. Therefore, 3/4 is one part when 3 is divided into 4 equal parts--which is one interpretation of 3 ÷ 4. (This latter interpretation of division is often called "equal shares" or "partitive.")

5.5b Understand how to divide a fraction by a fraction and to solve related problems.

- As with whole numbers, division of fractions is just a different way to write multiplication: if *A*, *B*, and *C* are fractions with *B* ≠ 0, then *A/B = C* means *A = C • B*

  Note: The dot (•) is an alternative to the cross (x) as a notation for multiplication. (Computers generally use the asterisk (*) in place of a dot.) In written mathematics, but never on a computer, the dot is often omitted (e.g., *ab* means a•b). As students move beyond the arithmetic of whole numbers to the arithmetic of fractions and decimals, the symbols • and / tend to replace x and ÷.

- Divide a fraction *a/b* (where *b*≠0) by a non-zero whole number *c*: because $\frac{a}{b} = \frac{a}{bc} \times c$, this division follows the rule $\frac{a/b}{c} = \frac{a}{bc}$.

  Example: $\frac{6/7}{4} = \frac{6}{4 \times 7}$ because $\frac{6}{7} = \frac{6}{4 \times 7} \times 4$. In the partitive interpretation of division, $\frac{6/7}{4}$ is one part in a division of $\frac{6}{7}$ into 4 equal parts.

- Divide a whole number *a* by a unit fraction $1/b$ ($b \neq 0$): because $a = ab \times {}^1/_b$, this division follows the rule $a/({}^1/_b) = ab$.

  Example: Because $5 = (5 \times 6) \times {}^1/_6$, $5/({}^1/_6) = 5 \times 6$. in the measurement sense of division, $5/({}^1/_6) = 5 \times 6$ is the answer to the question "how many parts of size ${}^1/_6$ can 5 be divided into?" Since there are 6 parts of size ${}^1/_6$ in one whole, there are 5×6 parts of size ${}^1/_6$ in 5 wholes.

5.5c Express division with remainder in the form of mixed numbers.

- When a division problem $a \div b$ is resolved into a quotient *q* and a remainder *r*, then $a = q \times b + r$. It follows that a/b equals the fraction (qb +r)/b, which in turn equals the mixed number = $q{}^r/_b$. .

  Example: $\frac{37}{7} = \frac{(5\times7)+2}{7} = \frac{(5\times7)}{7} + \frac{2}{7} = 5 + \frac{2}{7}$, which is equal to $5\frac{2}{7}$ by definition.

  Note: Fractions greater than 1 are often called *improper* fractions, although there is no justification nor need for this label.

5.5d Understand division as the inverse of multiplication, and *vice versa*.

  Note: Division was defined in Grade 2 as an action that reverses the results of multiplication. At that time, using only integers, division was limited to composite numbers and their factors (e.g., $6 \div 3$, but not $6 \div 4$). Only now, using fractions as well as whole numbers, can this inverse relationship be fully understood.

  Note: Although in previous grades the word number meant positive whole number, hereafter it will generally mean positive fraction, which encompasses all whole and mixed numbers.

- For any numbers *a* and *b* with $b \neq 0$, $(a \times b) \div b = a$ and $(a \div b) \times b = a$. In words, if a number (fraction) *a* is first multiplied by *b* and then divided by *b*, the result is the original number *a*, and the same is true if we first divide and then multiply.

5.6 Understand how to multiply terminating decimals by whole numbers.

5.6a Multiplying a terminating decimal by a whole number is equivalent to multiplying a fraction by a whole number.

  Example: 7.53 x 5 = (753/100) x 5 = (753 x 5) /100 = 3765/100 = 37.65.

5.6b Understand how to place the decimal point in an answer to a multiplication problem both by estimation and by calculation.

  Example. $5 \times 0.79 = 3.95$ because $5 \times \frac{79}{100} = \frac{5 \times 79}{100} = \frac{395}{100} = 3.95$. This can easily be estimated because 0.79 is less than 1, so 5×0.79 must be less then 5. Therefore the answer cannot be 395.0 or 39.5. Similarly, since $5 > 1$, 5×0.79 must be greater than .79, so the answer cannot be .395. Thus it must be 3.95.

- When a number is multiplied by a power of ten, the place value of the digits in the number are increased according go the power of ten; the reverse happens when a number is divided by a power of ten.

  Note: As a consequence, when multiplying a whole number by 10, 100, 1,000, the decimal point shifts to the right by 1, 2, or 3 places. Similarly, when dividing a whole number by 10, 100, 1,000, the decimal point shifts to the left.

5.6c Demonstrate with examples that multiplication of a number by a decimal or a fraction may result in either a smaller or a larger number.

  Note: Decimals, like fractions, can be greater than one.

5.7 Understand the notation and calculation of positive whole number powers.

5.7a Recognize and use the definition and notation for exponents.

- If p is a positive whole number, then a^p means $a \times a \times a \times \ldots \times a$ (p times).

  Note: Emphasize two special cases: powers of 2 and powers of 10.
- Understand and use the language of exponents and powers.

  Note: In the expression 10^3, 3 is an *exponent*, and 10^3 is a *power* of 10.

5.8 Solve multi-step problems using multi-digit positive numbers, fractions, and decimals.

5.8a Solve problems of various types--mathematical tasks, word problems, contextual questions, and "real-world" settings.

Note*:* As noted earlier, problem-solving is an implied part of all expectations. To focus on strategies for solving problems that are cognitively more complex than those previously encountered, computational demands should be kept simple.

5.8b Translate a problem's verbal statements or contextual details into diagrams, symbols, and numerical expressions.

5.8c Express answers in appropriate verbal or numerical form.

- Provide units in answers.
- Use estimation to judge reasonableness of answers.
- Use calculators to check computations.
- Round off answers as needed to a reasonable number of decimal places.

5.8d Solve problems that require a mixture of arithmetic operations, parentheses, and arithmetic laws (commutative, distributive, associative).

5.8e Use mental arithmetic with simple multiplication and division of whole numbers, fractions, and decimals.

Number & Operations

6.1 Understand and use negative numbers.

6.1a Know the definition of a negative number and how to locate negative numbers on the number line.

- If a is a positive number, $-a$ is a number that satisfies $a + (-a) = 0$.
- On the number line, $-a$ is the mirror image of a with respect to 0; it lies as far to the left of 0 as a lies to the right.

  Note: In elementary school, a negative number $-a$ is sometimes called the "opposite" of a, but this terminology is not used in later grades.
- Negative numbers may be either whole numbers or fractions.
- The positive whole numbers together with their negative counterparts and zero are caller *integers.*

  Note: The properties of negative numbers apply equally to integers and to fractions. Thus it is just as effective (and certainly easier) to limit to integers all examples that introduce the behavior of negative numbers.

  Note: The positive fractions together with the negative fractions and zero(which include all integers) are called *rational numbers*. In grade 6, these are all the numbers we have, so they are usually referred to just as "numbers." Later when irrational numbers are introduced, the distinction between rational and irrational will be important--but not now.

6.1b Understand why $-(-a) = a$ for any number a, both when a is positive and when a is negative.

- Use parentheses as in $-(-a)$ to distinguish the subtraction operation (minus) from the negative symbol.

6.1c Use the number line to demonstrate how to subtract a larger number from a smaller one.

- If $b > a$, the point c on the number line that lies at distance $b-a$ to the left of zero satisfies the relation $a - b = c$. Thus $a - b = -(b-a)$.
- Subtracting a smaller from a larger number is the same as adding the negative of the smaller number to the larger. That is, if $a > b$ then $a - b = a + (-b)$.

  Note: Formally, $a + (-b) = (a - b) + b + (-b) = (a - b) + 0 = a - b$.
- Recognize that $a + (-b) = a - b$ (even when $b > a$).

  Example: $3 - 8 = -5$ because $5 + (3 - 8) = 5 + (3 + (-8)) = (5 + 3) + (-8) = 8 + (-8) = 0$. Therefore, $3 - 8$ satisfies the definition of -5 as being that number which, when added to 5, yields zero.

6.1d Recognize that all numbers, positive and negative, satisfy the same commutative, associative and distributive laws.

  Note: Demonstrations of these laws are part of Algebra, below. Here recognition and fluent use are the important issues.

6.2 Understand how to divide positive fractions and mixed numbers.

6.2a. Understand that division of fractions has the same meaning as does division of whole numbers.

- Just as $A/B = C$ means that $A = C \times B$ for whole numbers, so $(a/b) / (c/d) = M/N$ means that $(a/b) = (M/N) \times (c/d)$.

6.2b Use and be able to explain the "invert and multiply" rule for division of fractions.

- Invert and multiply means: (a/b)/(c/d) = (a/b)x(d/c) = (ad)/(bc).

  Note: To verify that (a/b)/(c/d) = (ad)/(bc), we need to check that (ad)/(bc) satisfies the definition of (a/b)/(c/d), namely, that (a/b) = (ad)/(bc) x (c/d): (ad)/(bc) x (c/d) = (a/b)x(d/c) x (c/d) = (a/b) x 1 = a/b.

6.2c Find unknowns in division and multiplication problems using both whole and mixed numbers.

- Solve problems of the form $a \div [\,] = b$, $a \times [\,] = b$, $[\,] \div a = b$.

  Examples: $\frac{1}{4} \div [\,] = 1$; $\frac{1}{4} \div [\,] = \frac{3}{4}$; $\frac{1}{2} = 1 \times [\,]$; $2\frac{1}{3} \div 1\frac{1}{2} = [\,]$; $2\frac{1}{3} \div [\,] = 1\frac{1}{2}$.

6.2d Create and solve contextual problems that lead naturally to division of fractions.

- Recognize that division by a unit fraction $1/n$ is the same as multiplying by its denominator n.

6.3 Understand and use ratios and percentages.

6.3a Understand ratio as a fraction used to compare two quantities by division.

- Recognize *a:b* and a/b as alternative notations for ratios.

  Note: A ratio is often thought of as a pair of numbers rather than as a single number. Two such pairs of numbers represent the same ratio if one is a non-zero multiple of the other or equivalently, if when interpreted as fractions, they are equivalent.

  Example: 2:4 is the same ratio as 6:12, 8:16,or 1:2.
- Understand that quantities a and b can be compared using either subtraction (a-b) or division (a/b).
- Recognize that the terms *numerator* and *denominator* apply to ratios just as they do to fractions.

6.3b Understand that percentage is a standardized ratio with denominator 100.

- Recognize common percentages and ratios based on fractions whose denominators are 2, 3, 4, 5 or 10.

  Examples: 20%, 25%, $33\frac{1}{3}$%, 40%, 50%, $66\frac{2}{3}$% 90%, and 100% and their ratio, fraction, and decimal equivalents.
- Express the ratio between two quantities as a percent, and a percent as a ratio or fraction.

6.3c Create and solve word problems involving ratio and percentage.

- Write number sentences and contextual problems involving ratio and percentage.

6.4 Understand and use exponents and scientific notation.

6.4a Calculate with integers using the law of exponents: $b^n \times b^m = b^{n+m}$.

- Just as $b \times n$ (with b and n positive) can be understood as b added to itself n times, so b^n can be understood as b multiplied by itself n times.

  Note: The law of exponents for positive exponents is just a restatment of this definition, since both $b^n \times b^m$ and b^{n+m} mean b multiplied by itself $n+m$ times.

  Note: If $b > 0$, $b^1 = b$, $b^0 = 1$. The same is true of $b < 0$. If $b=0$, $0^1=0$, but 0^0 is not defined.
- If $n > 0$, b^{-n} means $1/b^n$ (that is, 1 divided by b n times).

Note: This definition of b^{-n} is designed to make the law of exponents work for all integers (positive or negative): $b^n \times b^{-n}$ means b multiplied by itself n times, then divided by *b n* times, yielding 1. Thus $b^n \times b^n = b^{n+(-n)} = b^0 = 1$.

Example: $3^{-2} = (1/3)^2 = 1/9$; $3^3 = 27$; $3^3 \times 3^{-2} = 3^{(3-2)} = 3^1 = 3 = 27 \times 1/9 = 27/9$.

6.4b Understand scientific notation and use it to express numbers and to compute products and quotients.

- Recognize the importance of scientific notation to express very large and very small numbers.
- Locate very large and very small numbers on the number line.
- Understand the concept of *significant digit* and the role of scientific notation in expressing both magnitude and degree of accuracy.

6.4c Model exponential behavior with contextual illustrations based on population growth and compound interest.

6.5 Solve multi-step mathematical, contextual and verbal problems using rational numbers.

6.5a Solve arithmetic problems involving more than one arithmetic operation using rational numbers.

- Calculate with and solve problems involving negative numbers, percentages, ratios, exponents, and scientific notation.

  Note: As usual, keep calculations simple in order to focus on the new concepts.
- Compare numbers expressed in different ways and locate them on the number line.
- Write number sentences involving negative numbers, percentages, ratios, exponents, and scientific notation.

6.5b Solve relevant contextual problems (e.g., sports, discounts, sales tax, simple and compound interest).

- Represent problems mathematically using diagrams, numbers, and symbolic expressions.
- Express answers clearly in verbal, numerical, symbolic, or graphical form.
- Use estimation to check answers for reasonableness and calculators to check for accuracy.
- Describe real situations that require understanding of and calculation with negative numbers, percentages, ratios, exponents, and scientific notation.

Number & Operations

7.1 Understand and work with the system of rational numbers.

7.1a Understand the meaning of rational numbers.

- Rational numbers are numbers that can be expressed as a ratio of integers.
- Whole numbers are rational numbers since they can be expressed as a ratio with the integer 1 in the denominator.
- Mixed numbers are rational numbers since they can be expressed as a so-called "improper" fraction.
- Finite (terminating) decimal numbers are rational numbers.
- Negative fractions are rational numbers since they can be expressed as the ratio of a negative and a positive integer.

  Note: Although this may seem self-evident, it is actually rather subtle. The rational number −12/7 is not defined as a fraction, but as the "opposite" of 12/7, i.e., −12/7 + 12/7 = 0. As a fraction, the positive rational number 12/7 twelve one-sevenths of a whole. But no similar interpretation makes sense for −12/7. Nonetheless, we can show that −12/7 is indeed the quotient of −12 divided by 7 (just as it appears). Since 7 × (−12/7) means −12/7 added to itself 7 times, it can be represented as $-(\frac{12}{7}+\frac{12}{7}+\frac{12}{7}+\frac{12}{7}+\frac{12}{7}+\frac{12}{7}+\frac{12}{7}) = -((7 \times 12)/7) = -12$. Therefore, if $n = -12/7$, $7 \times n = -12$. This means, by the definition of division, that the rational number n is the fraction -12/7.
- Percents can be thought of as rational numbers since they are ratios with denominators equal to 100.

7.1b Locate rational numbers on the number line.

- Recognize that a number and its negative are mirror images with respect to the 0.
- Understand that between every two rational numbers, no materhow close together, there are many others.

  ***Note**:* One way to find some is to average the two given numbers, and then average the averages.

  ***Note**:* The visual image of the number line provides an anchor for understanding numerical size and order that is essential to the development of arithmetic intuition.

7.1c Understand the properties of rational numbers.

- Adding, subtracting, multiplying, and dividing rational numbers always produces another rational number.
- The sum and product of two rational numbers (positive or negative, integer or fraction) satisfies the commutative, associative and distributive laws.
- If *r* is a rational number, then r+0 = r and r x 1= r.
- If *r* and *s* are a rational numbers with r+s = 0, then s = -r (the *negative* of r).
- If *r* and *s* are a rational numbers with r × s = 1 and r ≠ 0, then s = 1/r (the *reciprocal* of r).

7.1d Understand and use standard rules for inequalities when comparing rational numbers.

- For any rational numbers a, b, and c,
  -- a < b implies $a + c < b$ + c;
  -- $a < b$ implies $-a > -b$;
  -- if c > 0, a < b implies ac < bc;

-- if c < 0, a < b implies ac < bc;.

Note: Numerical and number line examples are more important than literal formulas as demonstrations of these rules.

7.1e Understand and work with greatest common divisors and least common multiples.

- Recognize common abbreviations such as gcd and lcm.
- Use greatest common divisors to reduce fractions n/m and ratios n:m to an equivalent form in which gcd (n,m) = 1. Fractions in which gcd (n,m) = 1 are said to be in *lowest terms*.

7.1f Employ effective methods of calculation with rational numbers.

- Transform numbers from one form to another (fractions, decimals, percents, mixed numbers) so as to permit efficient calculation.
- Make judicious use of calculators or computers.
- Recognize the phenomenon of roundoff error and compensate for inaccuracies it introduces.

  Note: Much of the early manipulation in each problem should be performed manually, reserving calculator use to the final stage where a complex answer may need to be converted into decimal form. If a calculator is used too early, important information about the numerical character of the final answer may be lost to early roundoff error.
- Check answers both by estimation and by appropriate independent calculations.

  Examples: Multiply to check division; use decimals to check calculations with fractions; add a column of numbers up to check addition first done from top down.
- Use calculators to check manual computations, and use estimation to check calculator answers.

7.2 Understand methods for converting between fraction and decimal forms of rational numbers.

7.2a Understand how to convert a rational number into a decimal.

- The decimal form of a rational number either terminates or eventually repeats.

  Note: The remainder in long division is always less than the divisor. So if it does not terminate, the division algorithm will eventually return some remainder for a second time. Thereafter, the process will repeat *ad inifinitum*.
- A fraction has a terminating decimal expansion if and only if its denominator in reduced form has only 2s and 5s as factors.

  Note: This assertion is an example of a mathematical statement of considerable significance--the "if and only if" condition--that will be appear often in Grades 7 and 8.

  Note: The *reduced form* of a fraction is an equivalent fraction in which no factor is common to both the numerator and denominator.

  Examples: 15.4 = 154/10 = 77/5; 2.35 = 235/100 = 47/20.

7.2b Understand how to convert a finite decimal into a rational number.

- A terminating decimal equals a fraction with a denominator of the form 10^n where *n* is the number of digits in the number to the right of the decimal point.

  Note: Typically this fraction is reduced to an equivalent form by eliminating factors that are common to both the numerator and denominator (e.g., .68 = 68/100 = 17/25).

Note: Converting a repeating decimal to a fraction requires a bit of algebra, so is better left to a later grade.

7.2c Know divisibility rules and use them to help factor numbers.

- If N is a positive integer, and
  -- if the last digit is even, N is divisible by 2;
  -- if the sum of the digits of N is divisible by 3, so is N;
  -- if the last two digits of N form a number divisible by 4, so is N;
  -- if the last digit of N is a 5 or a 0, N is divisible by 5;
  -- if the sum of the digits of N is divisible by 9, N is also; and
  -- if the last digit of N is 0,N is divisible by 10.

  Example: 165 is divisible by both 3 (since 1+6+5 = 12 is divisible by 3) and by 5 (since 165 ends in 5). Thus 165= 3•5•11.

  Example*:* The divisibility rule for 3 follows from the fact that when a power of 10 is divided by 3 it has remainder 1. For example, 24 is divisible by 3 because

  24/3 = $2\bullet\frac{10}{3}+\frac{4}{3} = 2\bullet(3+\frac{1}{3})+\frac{4}{3} = 2\bullet 3+2\bullet\frac{1}{3}+\frac{4}{3} = 6+\frac{2}{3}+\frac{4}{3} = 6+\frac{2+4}{3}$.

  Therefore, $\frac{24}{3}=6+\frac{2+4}{3}$, so 24 is divisible by 3 because 2 + 4 is divisible by 3.

7.3 Understand and work with square and cube roots.

7.3a Understand the definition of a *root* of rational number.

- If $a^n = b$, so that b is a *power* of a, then *a* is said to be a *root* of *b*.

  Note: As subtraction undoes addition and division undoes multiplication, so roots undo powers.

- The most common root is the *square root* ,symbolized by √, corresponding to the second (square) power: If $a^2 = b$, then $a = \sqrt{b}$.
- If $a^3 = b$, then b is the *cube* of a, and a is the *cube root* of b.
- Know the squares of numbers from 1 to 12 and the cubes of numbers from 1 to 5.

  Note: Positive integers that are the squares of other integers are called *square numbers*. Knowing the squares of numbers from 1 to 12 carries with it knowledge of the square roots of square numbers between 1 and 144.

7.3b Recognize that unless they are integers, square, cube, and nth roots of integers are not rational numbers.

- Such roots are numbers that cannot be written as the ratio of two integers.

  Note: This is a consequence of the fact that integers can be uniquely factored into products of prime numbers. The actual proof is a bit subtle.

- Numbers that cannot written as the ratio of integers are called *irrational*. Square, cube, and nth roots provide the most common examples of irrational numbers.

7.3c Understand why $\sqrt{mn} = \sqrt{m}\bullet\sqrt{n}$ and why $(\sqrt{m})^2 = m$.

Note: This expected property merits special emphasis since it is about irrational numbers and therefore does not follow directly from earlier observations about rational numbers.

Note: To verify these properties, we use the defintion of root to transform the property to a statement about rational numbers--we already know to be true. Specifically, $(\sqrt{m}\bullet\sqrt{n})^2 = (\sqrt{m}\bullet\sqrt{n})\,(\sqrt{m}\bullet\sqrt{n}) = \sqrt{m}\bullet\sqrt{m}\bullet\sqrt{n}\bullet\sqrt{n} = (\sqrt{m})^2\bullet(\sqrt{n})^2 = mn$. Therefore $\sqrt{mn} = \sqrt{m}\bullet\sqrt{n}$.

7.3d Estimate square and cube roots and use calculators to find good approximations.

- If $a < n < b$, then $\sqrt{a} < \sqrt{n} < \sqrt{b}$.

  Example: Because $2.6^2 = 6.76$ and $2.7^2 = 7.29$, $\sqrt{7}$ is between 2.6 and 2.7. That is, since $6.76 < 7 < 7.29$, therefore $2.6 < \sqrt{7} < 2.7$.

- If an estimate *a* to $\sqrt{n}$ is too high (or too low), then the quotient b = n/a will be another estimate that is too low (or too high). In such a case, the average (a+b)/2 will be a better next estimate.

  Note: This method of successive approximations is used in many mathematical situations where exact answers are hard to calculate.

  Example: Since 7< 9, estimate that $\sqrt{7}$= 2.8. Since 7/2.8 = 2.5, this estimate is clearly too high. Next try (2.8+2.5)/2 = 2.65. Then 7/2.65 = 2.6415. Average once again, and test: (2.65+2.6415)/2 = 2.64575, and 7/2.64575 = 2.6457526... This give six significant digits in three repetitions of the process.

7.3e Use rational numbers (including percents) and roots to solve mathematical, contextual, and verbal problems.

Number & Operations

8.1 Understand the definition of irrational numbers and know some examples.

8.1a Name some common examples of irrational numbers and locate them approximately on the number line.

- Numbers that are not rational (i.e., those that cannot be expressed as the ratio of integers) are called *irrational* numbers.

  Examples: $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, and π are irrational numbers. (The number π is defined in the Geometry strand.)
- The decimal expansion of an irrational number never ends, and never repeats. (If it did, then the number would be rational.)
- Use the first few digits in the decimal expansion of an irrational number to locate it on the number line.
- Recognize that 22/7 and 3.14 are just approximations to the irrational number π.

  Note: When approximations are used in a calculation, the "squiggly" sign $\approx$ meaning "approximately" must be used rather than the equals sign =.

  Example: $(2 + \sqrt{2})(3 - \sqrt{2}) = 2\bullet 3 - 2\sqrt{2} + 3\sqrt{2} - 2 = 4 + \sqrt{2} \approx 5.4142$.

8.1b Use indirect arguments to show that certain numbers are irrational.

- Show that any non-zero rational multiple of an irrational number is irrational.

  Example: Suppose $(3/8)\sqrt{2}$ were rational. Then for some integers *m* and *n*, $(3/8)\sqrt{2} = m/n$. That would mean that $\sqrt{2} = (m/n)\bullet(8/3)$, which is a rational number. This $\sqrt{2}$ is actually irrational, this contradiction shows that our original supposition must be incorrect. Hence $(3/8)\sqrt{2}$ is irrational.
- Show that the square root of a positive integer is either an integer or irrational.

  Example: Suppose $\sqrt{2}$ were rational. Then $\sqrt{2}$ can be expressed as a fraction in lowest terms, i.e., $\sqrt{2} = m/n$ for some integers *m* and *n* where the greatest common divisor of *m* and *n* is 1. Thus $m^2 = 2n^2$. Since *n* can be factored uniquely into primes, n^2 must have an even number of prime factors. Hence $2n^2$, and thus m^2,must have an odd number of prime factors. But this cannot be true, since *m* also can be factored uniquely into primes, so m^2 must have an even number of prime factors. This contradiction shows that $\sqrt{2}$ cannot be rational.

  Note: This classic proof relies on two important properties of integers--the existence of a greatest common divisor, and the unique decomposition into primes.