Mathematics Achievement Partnership (MAP)

K–8 Mathematics Expectations

December 2004 Draft

Commentary

This December 2004 draft of the MAP mathematics expectations is a substantial revision of the May 2004 draft based on invited reviews received from several individuals and organizations. These notes focus primarily on issues raised by reviewers and provide rationale for my editing decisions.

MAP's* K–8 Mathematics Expectations *is a very complex document that is now in its fourth or fifth incarnation. Notwithstanding months of scrutiny, many errors and unclear explanations undoubtedly remain; almost certainly, through my editing, I have introduced some new difficulties as well. However, the goal of this draft is not perfection of detail, but to provide a version that responds to the most important concerns raised by reviewers. I hope I have succeeded in that objective.

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Organization

Structure. The May 2004 draft of MAP's *K–8 Mathematics Expectations* was arranged in nine chapters, one for each grade, K–8. Each chapter opened with four to five *Sample Problem Expectations* to illustrate what students should be able to do at the end of the year. This section was followed by a brief statement of *Major Goals* for the year, accompanied by *Expected Preparation*. The next page provided, in three parallel columns, a *Summary of Expectations* for the previous grade, the current grade and the subsequent grade, each classified into four strands of number and operation, measurement and data, geometry, and algebra. (This feature was especially popular with reviewers.) The summary was followed by the *Expectations* themselves, interspersed with illustrative problems. The final section was a *Problem Book* containing many additional problems, some relating to particular strands, others cutting across several strands.

This December 2004 draft is very different. It contains only the *Expectations*, without any of the introductory summaries or problems. (These other features will be added back into the document before it is published.) Moreover, to focus more readily on issues of mathematical development and consistency, and to provide a fresh perspective for reviewers, *this draft is organized by strand rather than grade*. This way, readers can trace the development of concepts from grade to grade within each of the four strands. Later, the same pages can be rearranged into a grade-level presentation.

Mathematics Expectations. This December 2004 draft is organized on four levels of expectations signaled by increasing left indents matched to decreasing font size. The first three levels, with numbers and bullets, contain student expectations stated in different degrees of detail. Items on these three levels contain statements of what students are expected to understand, know and be able to do. The fourth level, with the largest indent and smallest font size, is addressed to educators and devoted to comments that provide a context for the expectations. These are prefaced by the words *Example* or *Note* in bold italic.

Examples contain very brief illustrations to clarify the language of an expectation. They are not intended as examples of problems for students — those will be contained in the *Problem Book* associated with each grade — but merely as specific manifestations of general statements that appear in the expectations. *Notes* often contain proofs of statements in the expectations or describe links among different expectations; some contain cautions or suggestions for teachers. Since *Notes* are written for teachers, they often use vocabulary and concepts beyond the current grade level of student expectations.

Expectations given in the first three levels are organized in a hierarchy of decreasing generality and increasing specificity. The top-most level is typically broad ("Understand and use negative numbers") with various specifics enumerated below as numbered subitems or bullets ("Understand why −(−a) = a"). In most cases, subitems and bullets represent extensions or particulars that have some special importance, not a systematic elaboration of the general expectation under which they are situated.

Chunking. A distinctive feature of the MAP's *K–8 Mathematics Expectations* is that topics are addressed in significant chunks rather than spread uniformly across all grades. Although this approach has certain advantages, it created some alarm among reviewers who chaff at the absence of topics they expect to find in particular grades, especially those that provide a scaffold for future topics.

The issue of focused versus distributed topics is not so much about expectations as about curriculum. If, for example, a school district believes that students need to work on proportional relationships in grade 6 to be ready for linear equations in grade 7, it can certainly arrange its curriculum to provide that early preparation — even though the MAP expectation on proportional relationships does not appear until grade 7. Another example concerns the use of literal letters (x, y) for unknowns.

Some reviewers suggested that they should appear much earlier than they do and be used in every subsequent grade. This December 2004 draft retains the delayed appearance until they are part of a more substantial introduction to algebra. This way, the expectation can be met by teachers who prefer either early or late introduction.

Generalities

Progression. The goal of the MAP *K–8 Mathematics Expectations* is for students to progress from year to year in the several dimensions that constitute mathematical proficiency. These include the four main strands of mathematics used to organize these expectations (number, data, geometry, and algebra) as well as important cross-cutting goals such as reasoning, communicating, modeling and solving problems.

This draft primarily expresses mileposts without elaborating on how they are to be reached. Only in a few cases (e.g., fractions, similarity) does the draft illustrate details of expected progression. Several reviewers mentioned the apparent arbitrariness of where progression was stressed and where it seemed to be ignored. Some argued that the draft does not consistently stress the orderly progress necessary for children to acquire both skills and understanding.

Unfortunately, there is not room to convey details of progression in all areas in an overview document of this nature. State standards and district curriculum frameworks must fill in the details. The fact that progression is elaborated for some topics and not for others is not intended as a signal that progress is important in only a few topics. It is, however, intended as an illustration of how progress in every area can be properly anchored in mathematical thinking.

Scope and Pacing. The original goal of the exercise that produced MAP's *K–8 Mathematics Expectations* was to outline annual targets in K–8 education that are necessary to bring students to the performance level suggested in MAP's *Foundations for Success*, the 2001 publication that described what students should be able to know and do at the end of grade 8. In preparing this revision, I benchmarked these *K–8 Mathematics Expectations* against *Foundations for Success* to ensure that most expectations outlined in *Foundations* were at least approximated, if not fully covered, in this draft. Many minor items are different — some added, some omitted — but with only one exception the match generally is rather good. (That exception is "completing the square," a complex procedure with quadratic functions, which we had previously agreed to drop.)

By approximating the goals set forth in *Foundations for Success,* I knowingly ignored the advice of those reviewers who felt the goals were too ambitious. In most other areas, I sought diligently to take their advice. But here I resisted, for several reasons. First, the rapid pacing of the MAP project has been a well-known goal from the very beginning, so to abandon it would be, in effect, to abandon a fundamental rationale for the entire project. Second, there was some disagreement among reviewers about the ages at which children can learn different topics, with an experienced minority of both teachers and mathematicians arguing that some MAP pacing was in fact too slow. Third, most reviewers who argued that MAP included too much too fast based their analysis on experience with curricula that are packed with ancillary topics not included in MAP, so it is plausible to imagine that a good teacher working with a more focused curriculum could indeed cover more ground. Finally, the fundamental benefit of MAP is not about rushing to meet all of its expectations by grade 8 but about ensuring that all students see and learn mathematics from the coherent mathematical perspective provided by MAP's *Expectations*. It would be counterproductive to omit topics that are crucial to mathematical understanding just because some students — perhaps even many — will require more time to master them.

Means versus Ends. To the extent practicable, these expectations are intended to set goals for student learning and performance at the end of each grade. They are not designed to enumerate all of the steps required for students to achieve these year-end objectives. This singular focus on ends rather than means caused distress among many reviewers who saw the previous draft as incomplete in its coverage of necessary instructional topics.

For example, several reviewers expressed concern that factoring quadratics (in order to solve quadratic equations) is an expectation, yet factoring as a concept or method is not previously introduced. As a skill, factoring is certainly as difficult as many others that are given considerable emphasis, and any curriculum that leads students to success in solving quadratic equations by factoring would need to devote considerable time to mastering the art of factoring. For MAP, however, factoring is only a tool used in solving quadratic equations. It is not — at least not in grade 8 — a separate expectation that would focus on factoring as a general concept and skill.

Factoring is one of many examples of an important difference between a set of expectations and a curriculum framework. *Not everything that may need to be taught and learned to meet a MAP expectation is itself an expectation.* Many topics in school mathematics are means to an end, not ends in themselves. Such topics may be missing from the MAP expectations, but nonetheless must be present in any curriculum that hopes to meet the MAP expectations.

Problem Solving. A number of reviewers were frustrated with the May 2004 draft's approach to problem solving. Sometimes problem solving was listed explicitly as part of (or in addition to) an expectation, other times not; sometimes problem solving was restricted to "word problems," other times not; sometimes mathematical problems were included under the umbrella of problem solving, other times problem solving appeared to imply only contextual or word problems; etc.

There is really no dispute about solving problems: *Students should be expected to solve problems of all types using all of the mathematics they study*. The confusion is solely about rhetoric: Should expectations repeat the admonition about solving problems over and over again with every expectation (" ... and use it to solve problems"), or just include occasional special sections devoted to solving problems, or perhaps just highlight problem solving in a preface that overlays the entire set of expectations? Constant repetition dulls the impact of the expectation, while a note about problem solving isolated in a preface that is disconnected from each grade risks being ignored. Generally, I sought the middle ground: Whenever reasonable, I consolidated expectations about problem solving into occasional special sections, but with the understanding that this expectation should apply universally.

Problem Complexity. Problem solving is a skill that cuts across strands of number, data, geometry and algebra. As these strands grow in sophistication, the challenge of solving problems increases in cognitive complexity — from straightforward single-step problems to intricate multistep problems in which problem formulation is as much a challenge as problem solution. A general guideline mentioned occasionally in these expectations is worth repeating here: At each grade level, *students should be expected to solve problems that advance in either mathematical content or problem complexity, but not both at once*. In other words, problems requiring new strategies should use calculations that had been mastered earlier, and problems requiring new calculations should use strategies that had been mastered earlier. After each step has been taken separately, then they can be done together — generally in the next grade.

At first glance, this observation appears to be purely pedagogical, something that the MAP *K–8 Expectations* claim to avoid. The goal is to solve complex problems using complex mathematics, and it should not matter how this goal is achieved. It does matter, however, if readers assume that students need to meet all grade-level expectations *at the same time* and incorrectly infer from that assumption that the MAP expectations are impossibly demanding. *Integration of separate expectations into coherent mathematical proficiency — what some have called "profound understanding" — almost always lags individual skills by a year or more.*

Language

Verbs. In earlier drafts of the MAP *K–8 Mathematics Expectations*, the meaning of and distinctions among different verbs created a great deal of confusion. Not only were reviewers critical of seemingly random differences in the nature of verbs used in different expectations ("understand" or "know" or "explain" or "be able to"), but even MAP advisers interpreted these action words in very different ways.

A generic example can be seen in the following list of possible ways to express a single expectation:

- Understand that the product of two negative numbers is positive.
- Understand why the product of two negative numbers is positive.
- Know that the product of two negative numbers is positive.
- Know how to multiply two negative numbers.
- Use the fact that the product of two negative numbers is positive to solve problems.
- Be able to explain why the product of two negative numbers is positive.

Often two or more verbs are combined in a single expectation, as in "understand and use" or "know and be able to."

Having failed in several previous attempts to clarify the ambiguity surrounding these verbs, in this draft I offer a different approach: simply state the mathematics ("the product of two negative numbers is positive") without attempting to dictate the desired state of students' minds (understanding, knowledge, memory, skill) regarding this fact. I adopt this strategy whenever it seems convenient and practical, but not universally. There are still plenty of ambiguous verbs left in these expectations — just fewer than in earlier drafts.

Adverbs. Earlier drafts of these expectations were full of advice about how the various verbs were to be carried out, which drew criticism from some reviewers. For example, expectations to "divide efficiently" and "find quickly" made some reviewers imagine a regimen of memorization, while exhortations to "use appropriately" or "understand fully" made others struggle to discern the intended meaning of the added adverb.

In this December 2004 draft, I have removed a number unnecessary adjectives — for example I deleted a few admonitions to do things "correctly" (how else?). I also continue the campaign against redundancy, by removing "be able to" (can one imagine a standard that began "be unable to"?).

I did, however, retain those that made a mathematical point (e.g., "divide mentally," "calculate manually," "record systematically," "estimate accurately"). In addition, when there were no obvious or agreed upon fixes, I let the problematic adverbs remain for the time being.

Nouns. In contrast to the persistent vagueness of verbs concerning students' thoughts and actions, the MAP *K–8 Mathematics Expectations* make a special effort to define mathematical objects and relationships carefully, logically and consistently. The goal of this effort is not only to help students be clear about what words such as "ratio" or "parallel" really mean but also to illustrate throughout the K–8 program that careful definitions are an essential and distinguishing feature of mathematical thinking.

Most mathematical terms are defined in these *Expectations* without reference to students' knowledge or performance. Instead of saying that students should "know" or "understand" or "use" the definition of parallel lines, this draft says simply that parallel lines are lines that never meet no matter how far they are extended. We simply assume that students should know, understand and be able to use the mathematics outlined in these expectations.

Specifics

Fractions. A careful, methodical introduction to fractions is a distinctive feature of MAP's *K–8 Mathematics Expectations*. There is plenty of evidence that, under current practices, many students leave middle school permanently confused about fractions, ratios and percents. MAP's diagnosis is that this adult turmoil is due to confusing school interpretations that do not make mathematical sense and consequently impede understanding and encourage thoughtless memorization. In contrast, the MAP expectations concerning fractions build carefully and logically over several years, beginning with unit fractions that are interpreted as a part of a whole. Recognition that a fraction thus defined is indeed a number that can be situated on the number line follows, but only with careful explanation and numerous examples.

Critics among the MAP reviewers argue that it is a mistake to introduce fractions well before children have mastered the arithmetic of whole numbers. They claim that premature emphasis on computational proficiency with fractions leads to superficial learning that short-changes conceptual understanding. According to this view, MAP's goal of deep conceptual understanding would more likely be achieved by focusing on whole numbers in the early grades and deferring the subtleties of fractions until the middle grades.

Only comparative classroom trials can determine whether early or late introduction of fractions is better — or if it really makes any difference at all. What is most important about the MAP approach to fractions is not its timing but its logical sequencing. The same benefits of careful development would accrue if the introduction were delayed one or two years (for instance, by trading curricular space with some parts of geometry or data currently suggested for later grades).

Number Line. Another distinctive feature of the MAP *K–8 Mathematics Expectations* is the consistent and strong emphasis on the number line as both an anchor and metaphor for the central subject of these expectations: number. The number line provides a consistent and mathematically accurate visual representation of numbers, no matter what confusing form their symbolic expressions may take, including whole numbers, fractions, roots, mixed numbers, decimals, ratios, percents and pi. In appearance these symbols present a bewildering display of notation (3, $^2/_5$, $\sqrt{3}$, $2^3/_7$, 2.35, 3:5, 23.5%, π). The number line unifies these varied formats in a profound mental image that will carry students well into the study of calculus and beyond.

Some reviewers expressed concern that the number line was overemphasized at the expense of other options (e.g., manipulatives) and that, in any case, it was inappropriate for the very early grades when children think of numbers as counting numbers rather than as distances along a ruled line. To respond to the latter concern, in the early grades I have used the phrase "discrete number line" to signify the image of whole numbers lined up in a row, evenly spaced, but without any continuous line joining them. The subsequent transition to the continuous line in grades 2 or 3 is straightforward.

The issue of manipulatives as possible alternatives to the number line is more complicated. Manipulatives are tools designed to help students learn mathematics and should be used whenever teachers find them appropriate. In contrast, *the number line is part of mathematics itself*. It is like a triangle or circle — an abstract object with subtle and important properties that undergirds much of what occurs in mathematics. Whereas the MAP *K–8 Mathematics Expectations* say little about manipulatives because they are primarily of pedagogical interest, the expectations say a great deal about the number line because it is a mathematical object that must be studied and understood.

Measurement. Some reviewers expressed concern over the location of measurement topics, namely whether they should be with Data or Geometry. Some of this concern is caused by school tradition, some by the organization of the NCTM *Standards* and some by MAP's own shift from *Foundations for Success,* in which measurement was divided between Geometry and Data, to the *K–8 Mathematics Expectations,* in which the section on data was renamed Measurement and Data.

In reality, there are two very different kinds of measurement topics involved in K–8 mathematics. One, anchored in geometry, involves the logical development of general concepts of length, area and volume based on standard units and the relation of these concepts to the numerical operation of multiplication. The other, anchored in data, concerns real-world measurements with units such as inches, quarts, dollars, kilobytes or hours and involves such topics as conversion factors, measurement errors and derived measures (e.g., velocity or miles per gallon). To the extent feasible, in this revision I reintroduced that distinction (which also was present in *Foundations for Success*).

Calculators. Based on criticism from a few reviewers about the lack of clarity concerning the role of calculators, I inserted occasional bullets providing expectations that students should know how to use a calculator to check their work, to generate and observe numerical patterns, and occasionally to carry out complex calculations. The presence of these specific statements implies a constraint on the purpose and role of calculators in the K–8 mathematics program that should implicitly clarify MAP's intent that other expectations be fulfilled without calculators.

Of course, this "compromise" falls short of meeting the substantive concerns of reviewers who argue that calculators are an important everyday tool whose responsible use must be explicitly taught in school mathematics programs. In a famous paper about 20 years ago, AT&T Bell Labs mathematician Henry Pollack wrote that technology not only changes *how* we teach and learn mathematics but also *what* mathematics we should be teaching and learning. The items I inserted about calculators minimally address Pollack's first point, but to address his second point — which is what many MAP reviewers are concerned about — would require major rethinking of the mathematical priorities of *Foundations for Success*. *This I did not do*.

Patterns. Many reviewers criticized the May 2004 draft for undervaluing the importance of exploring patterns (numerical and geometric; increasing, decreasing, cyclic; linear, inverse, exponential; ...). Two reasons may be advanced for this situation. One, in school mathematics, exploring patterns is more often a pedagogical device than an outcome expectation. Because the *K–8 Mathematics Expectations* focus on student outcomes rather than teacher practices, it is not surprising that particular exploration strategies used by good mathematics teachers are not visible in these expectations.

The second reason is a bit more subtle but nonetheless important to understand. The kind of reasoning involved in detecting patterns is inferential, not deductive. Mathematicians employ both kinds of reasoning in their work. However, asking for the next few terms in the sequence 1, 2, 4, 8, 16, ... is not definitively answerable since there are many logically possible extensions of this (or any) sequence. It might be 32, 64, 128, ... if the sequence represents powers of 2, or it might be 5, 10, 3, 6, ... if the sequence represents the end values of the famous "3n+1" puzzle. Instead of expecting students to read the mind of question-writers by knowing that powers of 2 are more likely than other possibilities, MAP expects students to be able to determine and extend patterns when a rule is given (e.g., name the first 10 terms in the sequence $f(n) = 2^{(n-1)}$).

Technicalities

Typography. Even an inattentive reader will easily notice that the typography in this draft is rather messy. The primary reason is that Word is a terrible editor for mathematics manuscripts, so there are many "work-arounds" embedded in the text. Many of these are graphic images of examples, especially of complex fractions, that were created early in the process by an external equation editor. Because these are fixed images, they cannot be easily edited and thus have remained unchanged throughout several revisions of the manuscript.

The same should be said about illustrations. For lack of skill and appropriate tools, I have made no attempt to add illustrations (except a few routine number-line graphics that can be created from a keyboard). So the illustrations are exactly those from the first version of these *Expectations*, even though there are several different expectations or examples that now merit different and additional graphics.

A third problem, slightly more subtle, is that Word knows nothing about the conventions of mathematical typing and typesetting in which letters used as variables are set in italic to distinguish them from regular text. Changing scores of individual letters to italic is a tedious chore that is only partially implemented in this draft.

As a draft intended for editorial and mathematical review, this manuscript is not greatly handicapped by these typographical infelicities. But they do distract from understanding in any public presentation and may well cause casual readers to worry that the logic of the document is as sloppy as its typography. It would be wise to prepare a more professional presentation before any wider distribution of the document.