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Algebra I • Module 3

Linear and Exponential Functions

OVERVIEW

In earlier grades, students define, evaluate, and compare functions and use them to model relationships between quantities (8.F.A.1, 8.F.A.2, 8.F.A.3, 8.F.B.4, 8.F.B.5). In this module, students extend their study of functions to include function notation and the concepts of domain and range. They explore many examples of functions and their graphs, focusing on the contrast between linear and exponential functions. They interpret functions given graphically, numerically, symbolically, and verbally; translate between representations; and understand the limitations of various representations.

In Topic A, students explore arithmetic and geometric sequences as an introduction to the formal notation of functions (F-IF.A.1, F-IF.A.2). They interpret arithmetic sequences as linear functions with integer domains and geometric sequences as exponential functions with integer domains (F-IF.A.3, F-BF.A.1a). Students compare and contrast the rates of change of linear and exponential functions, looking for structure in each and distinguishing between additive and multiplicative change (F-IF.B.6, F-LE.A.1, F-LE.A.2, F-LE.A.3).

In Topic B, students connect their understanding of functions to their knowledge of graphing from Grade 8. They learn the formal definition of a function and how to recognize, evaluate, and interpret functions in abstract and contextual situations (F-IF.A.1, F-IF.A.2). Students examine the graphs of a variety of functions and learn to interpret those graphs using precise terminology to describe such key features as domain and range, intercepts, intervals where the function is increasing or decreasing, and intervals where the function is positive or negative (F-IF.A.1, F-IF.B.4, F-IF.B.5, F-IF.C.7a).

In Topic C, students extend their understanding of piecewise functions and their graphs including the absolute value and step functions. They learn a graphical approach to circumventing complex algebraic solutions to equations in one variable, seeing them as \( f(x) = g(x) \) and recognizing that the intersection of the graphs of \( f(x) \) and \( g(x) \) are solutions to the original equation (A-REI.D.11). Students use the absolute value function and other piecewise functions to investigate transformations of functions and draw formal conclusions about the effects of a transformation on the function’s graph (F-IF.C.7b, F-BF.B.3). Note that the full treatment of F-BF.B.3, including recognizing even and odd functions, will occur in Algebra II, as defined by the PARCC Model Content Framework for Algebra I.

Finally, in Topic D students apply and reinforce the concepts of the module as they examine and compare exponential, piecewise, and step functions in a real-world context (F-IF.C.9). They create equations and functions to model situations (A-CED.A.1, F-BF.A.1, F-LE.A.2), rewrite exponential expressions to reveal and relate elements of an expression to the context of the problem (A-SSE.B.3c, F-LE.B.5), and examine the key features of graphs of functions, relating those features to the context of the problem (F-IF.B.4, F-IF.B.6).

The Mid-Module Assessment follows Topic B. The End-of-Module Assessment follows Topic D.
Focus Standards

Write expressions in equivalent forms to solve problems.

A-SSE.B.3  Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.*

   c.  Use the properties of exponents to transform expressions for exponential functions. For example the expression $1.15^t$ can be rewritten as $(1.15^{1/12})^{12t} = 1.012^{12t}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.

Create equations that describe numbers or relationships.

A-CED.A.1  Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*

Represent and solve equations and inequalities graphically.

A-REI.D.11  Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.*

Understand the concept of a function and use function notation.

F-IF.A.1  Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If $f$ is a function and $x$ is an element of its domain, then $f(x)$ denotes the output of $f$ corresponding to the input $x$. The graph of $f$ is the graph of the equation $y = f(x)$.

F-IF.A.2  Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

F-IF.A.3  Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. For example, the Fibonacci sequence is defined recursively by $f(0) = f(1) = 1$, $f(n+1) = f(n) + f(n-1)$ for $n \geq 1$.

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* Tasks have a real-world context. As described in the standard, there is an interplay between the mathematical structure of the expression and the structure of the situation such that choosing and producing an equivalent form of the expression reveals something about the situation. In Algebra I, tasks are limited to exponential expressions with integer exponents.

* In Algebra I, tasks are limited to linear, quadratic, or exponential equations with integer exponents.

* In Algebra I, tasks that assess conceptual understanding of the indicated concept may involve any of the function types mentioned in the standard except exponential and logarithmic functions. Finding the solutions approximately is limited to cases where $f(x)$ and $g(x)$ are polynomial functions.

* This standard is part of the Major Content in Algebra I and will be assessed accordingly.
Interpret functions that arise in applications in terms of the context.

F-IF.B.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; and end behavior; and periodicity.*

F-IF.B.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*

F-IF.B.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.*

Analyze functions using different representations.

F-IF.C.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.*

a. Graph linear and quadratic functions and show intercepts, maxima, and minima.

b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.

F-IF.C.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.

Build a function that models a relationship between two quantities.

F-BF.A.1 Write a function that describes a relationship between two quantities.*

a. Determine an explicit expression, a recursive process, or steps for calculation from a context.

* Tasks have a real-world context. In Algebra I, tasks are limited to linear functions, quadratic functions, square-root functions, cube-root functions, piecewise functions (including step functions and absolute-value functions), and exponential functions with domains in the integers.

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8 In Algebra I, tasks are limited to linear functions, quadratic functions, square-root functions, cube-root functions, piecewise functions (including step functions and absolute-value functions), and exponential functions with domains in the integers. The focus in this module is on linear and exponential functions.

9 Tasks have a real-world context. In Algebra I, tasks are limited to linear functions, quadratic functions, and exponential functions with domains in the integers.
Build new functions from existing functions.

F-BF.B.3\(^\text{10}\) Identify the effect on the graph of replacing \(f(x)\) by \(f(x) + k\), \(k f(x)\), \(f(kx)\), and \(f(x + k)\) for specific values of \(k\) (both positive and negative); find the value of \(k\) given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

Construct and compare linear, quadratic, and exponential models and solve problems.

F-LE.A.1 Distinguish between situations that can be modeled with linear functions and with exponential functions.\(^*\)

a. Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals.

b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.

c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.

F-LE.A.2\(^\text{11}\) Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).\(^*\)

F-LE.A.3 Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.\(^*\)

Interpret expressions for functions in terms of the situation they model.

F-LE.B.5\(^\text{12}\) Interpret the parameters in a linear or exponential function in terms of a context.\(^*\)

Foundational Standards

Work with radicals and integer exponents.

8.EE.A.1 Know and apply the properties of integer exponents to generate equivalent numerical expressions. For example, \(3^2 \times 3^{-5} = 3^{-3} = 1/3^3 = 1/27\).

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\(\text{10}\) In Algebra I, identifying the effect on the graph of replacing \(f(x)\) by \(f(x) + k\), \(k f(x)\), \(f(kx)\), and \(f(x + k)\) for specific values of \(k\) (both positive and negative) is limited to linear and quadratic functions. Experimenting with cases and illustrating an explanation of the effects on the graph using technology is limited to linear functions, quadratic functions, square-root functions, cube-root functions, piecewise functions (including step functions and absolute-value functions), and exponential functions with domains in the integers. Tasks do not involve recognizing even and odd functions. The focus in this module is on linear and exponential functions.

\(\text{11}\) In Algebra I, tasks are limited to constructing linear and exponential functions in simple (e.g., not multi-step) context.

\(\text{12}\) Tasks have a real-world context. In Algebra I, exponential functions are limited to those with domains in the integers.
8.EE.A.2 Use square root and cube root symbols to represent solutions to equations of the form \( x^2 = p \) and \( x^3 = p \), where \( p \) is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that \( \sqrt{2} \) is irrational.

**Define, evaluate, and compare functions.**

8.F.A.1 Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output.\(^{13}\)

8.F.A.2 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a linear function represented by a table of values and a linear function represented by an algebraic expression, determine which function has the greater rate of change.*

8.F.A.3 Interpret the equation \( y = mx + b \) as defining a linear function, whose graph is a straight line; give examples of functions that are not linear. *For example, the function \( A = s^2 \) giving the area of a square as a function of its side length is not linear because its graph contains the points \((1,1), (2,4)\) and \((3,9)\), which are not on a straight line.*

**Use functions to model relationships between quantities.**

8.F.B.4 Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two \((x, y)\) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.

8.F.B.5 Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.

**Reason quantitatively and use units to solve problems.**

N-Q.A.1 Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.

N-Q.A.2\(^{14}\) Define appropriate quantities for the purpose of descriptive modeling.

N-Q.A.3 Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.

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\(^{13}\) Function notation is not required in Grade 8.

\(^{14}\) This standard will be assessed in Algebra I by ensuring that some modeling tasks (involving Algebra I content or securely held content from Grades 6-8) require the student to create a quantity of interest in the situation being described.
Interpret the structure of expressions.

A-SSE.A.1 Interpret expressions that represent a quantity in terms of its context.*
   a. Interpret parts of an expression, such as terms, factors, and coefficients.
   b. Interpret complicated expressions by viewing one or more of their parts as a single entity.
      For example, interpret P(1+r)^n as the product of P and a factor not depending on P.

A-SSE.A.2 Use the structure of an expression to identify ways to rewrite it. For example, see x^4 – y^4 as
   \((x^2)^2 – (y^2)^2\), thus recognizing it as a difference of squares that can be factored as
   \((x^2 – y^2)(x^2 + y^2)\).

Create equations that describe numbers or relationships.

A-CED.A.2 Create equations in two or more variables to represent relationships between quantities;
   graph equations on coordinate axes with labels and scales.*

A-CED.A.3 Represent constraints by equations or inequalities, and by systems of equations and/or
   inequalities, and interpret solutions as viable or non-viable options in a modeling context. For
   example, represent inequalities describing nutritional and cost constraints on combinations of
   different foods.*

A-CED.A.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving
   equations. For example, rearrange Ohm’s law \( V = IR \) to highlight resistance \( R \).*

Understand solving equations as a process of reasoning and explain the reasoning.

A-REI.A.1 Explain each step in solving a simple equation as following from the equality of numbers
   asserted at the previous step, starting from the assumption that the original equation has a
   solution. Construct a viable argument to justify a solution method.

Solve equations and inequalities in one variable.

A-REI.B.3 Solve linear equations and inequalities in one variable, including equations with coefficients
   represented by letters.

Solve systems of equations.

A-REI.C.6\(^{15}\) Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on
   pairs of linear equations in two variables.

Represent and solve equations and inequalities graphically.

A-REI.D.10 Understand that the graph of an equation in two variables is the set of all its solutions plotted
   in the coordinate plane, often forming a curve (which could be a line).

\(^{15}\) Tasks have a real-world context. In Algebra I, tasks have hallmarks of modeling as a mathematical practice (e.g., less-defined tasks,
   more of the modeling cycle, etc.).
Focus Standards for Mathematical Practice

MP.1  **Make sense of problems and persevere in solving them.** Students are presented with problems that require them to try special cases and simpler forms of the original problem to gain insight into the problem.

MP.2  **Reason abstractly and quantitatively.** Students analyze graphs of non-constant rate measurements and apply reason (from the shape of the graphs) to infer the quantities being displayed and consider possible units to represent those quantities.

MP.3  **Construct viable arguments and critique the reasoning of others.** Students spend significant time developing conjectures about the effect, given \( f(x) \), of \( k \) on the graphs of transformed functions such as \( f(x + k) \) and \( f(kx) \). In the context of these endeavors students also respond to and critique the arguments and reasoning of others.

MP.4  **Model with mathematics.** Students have numerous opportunities to solve problems that arise in everyday life, society, and the workplace (e.g., modeling bacteria growth and understanding the federal progressive income tax system).

MP.6  **Attend to precision.** Students use precise language when describing the effect of various transformations on the graphs of functions. Students gain experience using the specialized language of transformations to describe graphs. Students learn to use descriptive statements such as, “The graph of \( f(x - 2) \) is the graph of \( f(x) \) translated two units to the right.”

MP.7  **Look for and make use of structure.** Students reason with and analyze collections of equivalent expressions to see how they are linked through the properties of operations. They discern patterns in sequences of solving equation problems that reveal structures in the equations themselves. (e.g., \( 2x + 4 = 10, 2(x - 3) + 4 = 10, 2(3x - 4) + 4 = 10 \)). Additionally, students come to see functions like \( g(x) = |x - 2| + 3 \) and \( h(x) = 2|x| \) as sharing structural similarities with \( f(x) = |x| \). Instead of seeing \( g(x) \) and \( h(x) \) as complicated expressions, students see them as related to or derived from the simpler \( f(x) \), and interpret the various values of \( k \) in the context of a graph.

MP.8  **Look for and express regularity in repeated reasoning.** After solving many linear equations in one variable (e.g., \( 3x + 5 = 8x - 17 \)), students look for general methods for solving a generic linear equation in one variable by replacing the numbers with letters (e.g., \( ax + b = cx + d \)). They pay close attention to calculations involving the properties of operations, properties of equality, and properties of inequalities, to find equivalent expressions and solve equations, while recognizing common ways to solve different types of equations. Additionally, after graphing many functions such as \( |x| + 2, |x| - 3, \) and \( |x| + 5 \), students are able to describe generally the effect of \( k \) on the graph of \( f(x) + k \).
Terminology

New or Recently Introduced Terms

- **Function** (A function is a correspondence between two sets, \(X\) and \(Y\), in which each element of \(X\) is matched\(^{16}\) to one and only one element of \(Y\). The set \(X\) is called the domain; the set \(Y\) is called the range.)
- **Domain** (Refer to the definition of function.)
- **Range** (Refer to the definition of function.)
- **Linear Function** (A linear function is a polynomial function of degree 1.)
- **Average Rate of Change** (Given a function \(f\) whose domain includes the closed interval of real numbers \([a, b]\) and whose range is a subset of the real numbers, the average rate of change on the interval \([a, b]\) is \(\frac{f(b)-f(a)}{b-a}\).)
- **Piecewise Linear Function** (Given non-overlapping intervals on the real number line, a (real) piecewise linear function is a function from the union of the intervals to the set of real numbers such that the function is defined by (possibly different) linear functions on each interval.)

Familiar Terms and Symbols\(^{17}\)

- Numerical Symbol
- Variable Symbol
- Constant
- Numerical Expression
- Algebraic Expression
- Number Sentence
- Truth Values of a Number Sentence
- Equation
- Solution
- Solution Set
- Simple Expression
- Factored Expression
- Equivalent Expressions
- Polynomial Expression
- Equivalent Polynomial Expressions
- Monomial
- Coefficient of a Monomial

\(^{16}\) “Matched” can be replaced with “assigned” after students understand that each element of \(x\) is matched to exactly one element of \(y\).

\(^{17}\) These are terms and symbols students have seen previously.
- Terms of a Polynomial

**Suggested Tools and Representations**

- Coordinate Plane
- Equations and Inequalities
- Graphing Calculator
### Assessment Summary

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Topic C:
Transformations of Functions

A-REI.D.11, F-IF.C.7b, F-BF.B.3

Focus Standard:

- A-REI.D.11: Explain why the x-coordinates of the points where the graphs of the equations \( y = f(x) \) and \( y = g(x) \) intersect are the solutions of the equation \( f(x) = g(x) \); find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where \( f(x) \) and/or \( g(x) \) are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.

- F-IF.C.7b: Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.
  - b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.

- F-BF.B.3: Identify the effect on the graph of replacing \( f(x) \) by \( (x + k) \), \( k f(x) \), \( f(kx) \), and \( f(x + k) \) for specific values of \( k \) (both positive and negative); find the value of \( k \) given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

Instructional Days: 6

Lesson 15: Piecewise Functions (E)
Lesson 16: Graphs Can Solve Equations Too (P)
Lessons 17–20: Four Interesting Transformations of Functions (E, P, E, E)

Lesson 15 of this topic formalizes the study of piecewise functions that began in Module 1. The study of piecewise functions in this lesson includes step functions and the absolute value function. Piecewise functions work nicely in the remaining lessons of this topic beginning with Lesson 16, where students learn that an equation \( f(x) = g(x) \), such as \( |x - 3| + 1 = |2x - 4| \), can be solved by finding the intersection points of the graphs of \( y = f(x) \) and \( y = g(x) \). Students use technology in this lesson to create the graphs and observe their intersection points. Next, in Lessons 17-20 students use piecewise functions as they explore four transformations of functions: \( f(x) + k, f(x + k), kf(x), \) and \( f(kx) \).

1 Lesson Structure Key: P-Problem Set Lesson, M-Modeling Cycle Lesson, E-Exploration Lesson, S-Socratic Lesson
Topic C focuses on opportunities for students to use and develop mathematical practices, particularly MP.3, MP.6, and MP.8. Students spend significant time developing conjectures about the effect, given \( f(x) \), of \( k \) on the graphs of the transformed functions, such as \( f(x + k) \) and \( f(kx) \). To this end, students are encouraged to use precise language when describing these effects, in addition to the appearance of graphs of functions in general, as in lesson 15. Students gain experience using the specialized language of transformations to describe graphs (e.g., “The graph of \( f(x - 2) \) is the graph of \( f(x) \) translated two units to the right.”) Additionally, after graphing many functions such as \( |x| + 2 \), \( |x| - 3 \), and \( |x| + 5 \), students are able to describe generally the effect of \( k \) on the graph of \( f(x) + k \). This process of developing generalizations occurs in lessons 17, 18, and 19.

Additionally, students develop their abilities with respect to MP.7, as, for example, students come to see functions like \( g(x) = |x - 2| + 3 \) and \( h(x) = 2|x| \) as sharing structural similarities with \( f(x) = |x| \). Instead of seeing \( g(x) \) and \( h(x) \) as complicated expressions, students see them as related to or derived from the simpler \( f(x) \), and interpret the various values of \( k \) in the context of a graph.
Lesson 15: Piecewise Functions

Student Outcomes

- Students examine the features of piecewise functions including the absolute value function and step functions.
- Students understand that the graph of a function \( f \) is the graph of the equation \( y = f(x) \).

Lesson Notes

This lesson has two main purposes: The first is to continue the work from Lessons 11–13 regarding the interplay between graphs, equations, and functions; the second is to expose students to piecewise functions in general and the absolute value and step functions, specifically. Lessons 12 and 13 established the meaning of the graph of a function and the graph of the equation \( y = f(x) \). This lesson continues to clarify that these two sets are one and the same. Students consider two important functions used in later lessons and classes: the absolute value function and the greatest integer function. The lesson focuses on standard F-IF.7b, as students graph piecewise-defined functions.

Use the optional Opening Exercise as an opportunity to informally assess student understanding of absolute value.

Classwork

Opening (2 minutes)

Recall that the absolute value of a number is the distance from 0 of a point on the number line. Because we are measuring distance, the absolute value of a nonzero number is always positive. For example, \(|-3| = 3 \) because the point \(-3\), located 3 units to the left of 0 on the real number line, is 3 units away from 0. Absolute value can also be used to define the distance between any two points on the real number line. For example, \(|5 - 8| = 3 \) because there are 3 units between the numbers 5 and 8 on the real number line.

Opening Exercise (3 minutes) (optional)

For each real number \( a \), the absolute value of \( a \) is the distance between 0 and \( a \) on the number line and is denoted \(|a|\).

1. Solve each one variable equation.
   a. \(|x| = 6\)  \(\{(-6, 6)\}\)
   b. \(|x - 5| = 4\)  \(\{(9, 1)\}\)
   c. \(2|x + 3| = -10\)  \(\{No solution.\}\)

2. Determine at least five solutions for each two-variable equation. Make sure some of the solutions include negative values for either \( x \) or \( y \).
   a. \(y = |x|\)  \(\{(-2, 2), (-1, 1), (0, 0), (1, 1), (2, 2)\}\)
   b. \(y = |x - 5|\)  \(\{(-1, 6), (0, 5), (1, 4), (5, 0), (6, 1)\}\)
   c. \(x = |y|\)  \(\{(1, 1), (1, -1), (0, 0), (2, 2), (2, -2)\}\)

Scaffolding:

- If students are struggling with these exercises, explicitly model how to find solutions.
- For students that may be ready for a challenge, pose the question, “Write three different absolute value equations that have a solution of \(\frac{1}{2}\). Explain any structural similarities they have.”
Exploratory Challenge 1 (15 minutes)

Have students work parts (a)–(d) in small groups. As you circulate, check to see that the groups are creating graphs. Remind them that the domain of the variables for these equations is all real numbers so their graphs should be continuous. Make sure groups are plotting (0,0) for parts (a) and (c) and (5,0) for part (b). After a few minutes, have different groups share their responses. Provide time for groups to revise their graphs as needed.

Part (d) offers an example of MP.6 as students must communicate their findings using precise language. A student example with particularly strong language may be highlighted for the benefit of the class.

For parts (a)–(c) create graphs of the solution set of each two-variable equation from Opening Exercise 2.

- a. \( y = |x| \)
- b. \( y = |x - 5| \)
- c. \( x = |y| \)
- d. Write a brief summary comparing and contrasting the three solution sets and their graphs.

Scaffolding:
- If students are struggling with these exercises, explicitly model how to create the graph in part a.
- Offer students graphing calculators or software to aid in creating the graphs.
- For students that may be ready, consider challenging them to graph equations such as \( y = -\frac{1}{2}|x| \) or \( x = |y - 1| + 4 \).

The graphs of parts (a) and (b) are the same except that part (b) has point of the ‘vee’ (the vertex of angle) at (5,0) instead of (0,0). The graph for part (c) looks like a 90° clockwise rotation of the graph from part (a) about the point (0,0). The points in the solution sets to parts (a) and (b) are a function, but the points in the solution set for part (c) are not.

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**COMMON CORE**

**Lesson 15:** Piecewise Functions

**Date:** 3/25/14

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The next portion asks students to consider their work so far in Module 3. Part (h) makes the connection that the graph of the equation \( y = |x| \) and the graph of \( f(x) = |x| \) are identical. Part (i) helps students understand that the graph of \( y = f(x) \) and the graph of a two-variable equation (in \( x \) and \( y \)) are only identical if the elements of the equation’s solution set define a function in \( x \) to begin with.

For parts (e) – (j) consider the function \( f(x) = |x| \) where \( x \) can be any real number.

e. Explain the meaning of the function \( f \) in your own words.

   This function assigns every real number to its absolute value, which is the distance the point is located from 0 on the real number line. Each number and its opposite will have the same range element. The number 0 will be assigned to 0.

f. State the domain and range of this function.

   Domain: all real numbers. Range: all non-negative real numbers.

g. Create a graph of the function \( f \). You might start by listing several ordered pairs that represent the corresponding domain and range elements.

h. How does the graph of the absolute value function compare to the graph of \( y = |x| \)?

   The two graphs are identical. They are identical because each ordered pair in the function would make the equation \( y = |x| \) a true number sentence if the domain value were substituted for \( x \) and the range value was substituted for \( y \). Therefore the graph of the function is the graph of the solution set of the equation.

i. Define a function whose graph would be identical to the graph of \( y = |x - 5| \)?

   Let \( g(x) = |x - 5| \) where \( x \) can be any real number.

j. Could you define a function whose graph would be identical to the graph of \( x = |y| \)? Explain your reasoning.

   No. The graph of \( x = |y| \) does not meet the definition of a graph of a function. If it were the graph of a function (say, the function \( h \)), it would be the set of ordered pairs \( \{(x, h(x)) \mid x \in D \} \), which means there would be only one \( y \)-value for each \( x \) in the domain \( D \). However, in the graph of \( x = |y| \) there is a number \( x \) (in fact, there are infinitely many \( x \)’s) associated with two different \( y \)-values: (3, 3) and (3, –3) are both solutions to the equation \( x = |y| \).
As you debrief questions (h)–(j) as a whole group, lead a discussion that includes a summary of the following information. When we create the graph of the solution set to a two-variable equation, we use essentially the same process as when we create the graph of \( y = f(x) \). We sift through all the \((x, y)\) pairs in the Cartesian plane and plot only those pairs that make a true number sentence. The difference between the two processes is that when we graph \( y = f(x) \), each \( x \) value in the domain of \( f \) will be paired with only one \( y \) value. When graphing a two-variable equation, there is no such restriction placed on the ordered pairs that return a true number sentence. The process of creating the graph of a function \( f \) yields the same results as graphing the solution set to the equation \( y = f(x) \) except we run through the set of domain values, determine the corresponding range value, and then plot that ordered pair. Since each \( x \) in the domain is paired with exactly one \( y \) in the range, the resulting graphs will be the same. For this reason, we often use the variable symbol \( y \) and the function name \( f(x) \) interchangeably when we talk about the graph of a function or two-variable equation solved for \( y \). The caveat is that the two-variable equation must have a solution set where each \( x \) is paired with only one \( y \).

### k. Let \( f_1(x) = -x \) for \( x < 0 \) and let \( f_2(x) = x \) for \( x \geq 0 \). Graph the functions \( f_1 \) and \( f_2 \) on the same Cartesian plane. How does the graph of these two functions compare to the graph in part (g)?

![Graph of two functions](image)

*The graph of these two functions when graphed on the same Cartesian plane is identical to the graph of the absolute value function.*

Close this portion of the lesson with the following definition of the absolute value function as a piecewise function.

### Definition:

The absolute value function \( f \) is defined by setting \( f(x) = |x| \) for all real numbers. Another way to write \( f \) is as a piecewise linear function:

\[
f(x) = \begin{cases} 
-x & x < 0 \\
 0 & x = 0 \\
 x & x > 0 
\end{cases}
\]
Example 1 (5 minutes)

This example shows students how to express a translation of the absolute value function as a piecewise function. Students create a graph of this function:

Let \( g(x) = |x - 5| \). The graph of \( g \) is the same as the graph of the equation \( y = |x - 5| \) you drew in the Exploratory Challenge 1, part (b). Use the redrawn graph below to rewrite the function \( g \) as a piecewise function.

Example 1

Explain that we will need to derive the equations of both lines to write \( g \) as a piecewise function.

Label the graph of the linear function with negative slope by \( g_1 \) and the graph of the linear function with positive slope by \( g_2 \) as in the picture above.

- Function \( g_1 \): Slope of \( g_1 \) is \(-1\) (why?), and the \( y \)-intercept is 5; therefore, \( g_1(x) = -x + 5 \).
- Function \( g_2 \): Slope of \( g_2 \) is 1 (why?), and the \( y \)-intercept is -5 (why?); therefore, \( g_2(x) = x - 5 \).

Writing \( g \) as a piecewise function is just a matter of collecting all of the different “pieces” and the intervals upon which they are defined:

\[
g(x) = \begin{cases} 
-x + 5 & x < 5 \\
5 & 5 \leq x < 8 \\
x - 5 & x \geq 8 
\end{cases}
\]

- How does this graph compare to the graph of the translated absolute value function?
  - The graphs are congruent, but the graph of \( g \) has been translated to the right 5 units. (Using terms like “congruent” and “translated” reinforces concepts from 8th grade and prepares students for geometry.)

- How can you use your knowledge of the graph of \( f(x) = |x| \) to quickly determine the graph of \( g(x) = |x - 5| \)?
  - Watch where the vertex of the graph of \( f \) has been translated. In this case, \( g(x) = |x - 5| \) has translated the vertex point from (0,0) to (5,0). Then, graph a line with a slope of 1 for the piece where \( x < 5 \) and a line with a slope of -1 for the piece where \( x > 5 \).

- Can we interpret in words what this function does?
  - The range values are found by finding the distance between each domain element and the number 5 on the number line.
Exploratory Challenge 2 (8 minutes)

This exploration introduces the two types of step functions and a third function that is related to them: the floor function (also known as the greatest integer function), the ceiling function, and the sawtooth function. The notation that one often sees for the greatest integer function is \( f(x) = \lfloor x \rfloor \). Gauss first introduced the greatest integer function in the early 1800s. Later, Iverson defined the floor and ceiling functions and introduced the notation you see below in 1962. Both notations are used in mathematics. These functions are used in computer programming languages among other applications. Be sure to explain the notation.

Parts (b) and (c) help students understand how the range values for each function are generated. In part (c), students will begin to understand that all real numbers in the interval have the same y-value. Clarify for students why the interval is closed at the left endpoint and open at the right endpoint. At this point in the lesson, consider asking students to attempt the challenge independently. If students are struggling to create graphs, you may need to finish this exploration as a whole class. Before closing the lesson, make sure each student has a correct graph of the functions.

**Exploratory Challenge 2**

The floor of a real number \( x \), denoted by \( \lfloor x \rfloor \), is the largest integer not greater than \( x \). The ceiling of a real number \( x \), denoted by \( \lceil x \rceil \), is the smallest integer not less than \( x \). The sawtooth number of a positive number is the “fractional part” of the number that is to the right of its floor on the number line. In general, for a real number \( x \), the sawtooth number of \( x \) is the value of the expression \( x - \lfloor x \rfloor \). Each of these expressions can be thought of as functions with domain the set of real numbers.

a. Complete the following table to help you understand how these functions assign elements of the domain to elements of the range. The first and second rows have been done for you.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \lfloor x \rfloor = \lfloor x \rfloor )</th>
<th>( \lceil x \rceil = \lceil x \rceil )</th>
<th>( \text{sawtooth}(x) = x - \lfloor x \rfloor )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.8</td>
<td>4</td>
<td>5</td>
<td>0.8</td>
</tr>
<tr>
<td>(-1.3)</td>
<td>(-2)</td>
<td>(-1)</td>
<td>0.7</td>
</tr>
<tr>
<td>2.2</td>
<td>2</td>
<td>3</td>
<td>0.2</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>(-3)</td>
<td>(-3)</td>
<td>(-3)</td>
<td>0</td>
</tr>
<tr>
<td>(\frac{2}{3})</td>
<td>(-1)</td>
<td>0</td>
<td>(\frac{1}{3})</td>
</tr>
<tr>
<td>(\pi)</td>
<td>3</td>
<td>4</td>
<td>(\pi - 3)</td>
</tr>
</tbody>
</table>

b. Create a graph of each function.

\[ \lfloor x \rfloor = \lfloor x \rfloor \]
\[ \lceil x \rceil = \lceil x \rceil \]
\[ \text{sawtooth}(x) = x - \lfloor x \rfloor \]
c. For the floor, ceiling, and sawtooth functions, what would be the range values for all real numbers \( x \) on the interval \([0, 1]\)? The interval \((1, 2]\)? The interval \([-2, -1]\)? The interval \([1.5, 2.5]\)?

- **Floor**: \([0]\), **Ceiling**: \([0, 1]\), **Sawtooth**: \([0, 1]\).
- **Floor**: \([1, 2]\), **Ceiling**: \([2]\), **Sawtooth**: \([0, 1]\).
- **Floor**: \([-2]\), **Ceiling**: \([-2, -1]\), **Sawtooth**: \([0, 1]\).
- **Floor**: \([1, 2]\), **Ceiling**: \([2, 3]\), **Sawtooth**: \([0, 1]\).

**Closing (2 minutes)**

Ask students to respond to summarize key points of the lesson, either in writing or with a neighbor.

- You can use different expressions to define a function over different subsets of the domain. These are called piecewise functions. The absolute value function and step functions can be represented as piecewise functions.
- The graph of a function \( f \) and the graph of the equation \( y = f(x) \) are the same.

**Relevant Vocabulary**

**Piecewise-Linear Function**: Given a number of non-overlapping intervals on the real number line, a **piecewise-linear function** is a function from the union of the intervals to the set of real numbers such that the function is defined by (possibly different) linear functions on each interval.

**Absolute Value Function**: The absolute value of a number \( x \), denoted by \(|x|\), is the distance between 0 and \( x \) on the number line. The **absolute value function** is the piecewise-linear function such that for each real number \( x \), the value of the function is \(|x|\).

We often name the absolute value function by saying, “Let \( f(x) = |x| \) for all real numbers \( x \).”

**Floor Function**: The floor of a real number \( x \), denoted by \([x]\), is the largest integer not greater than \( x \). The **floor function** is the piecewise-linear function such that for each real number \( x \), the value of the function is \([x]\).

We often name the floor function by saying, “Let \( f(x) = [x] \) for all real numbers \( x \).”

**Ceiling Function**: The ceiling of a real number \( x \), denoted by \([x]\), is the smallest integer not less than \( x \). The **ceiling function** is the piecewise-linear function such that for each real number \( x \), the value of the function is \([x]\).

We often name the ceiling function by saying, “Let \( f(x) = [x] \) for all real numbers \( x \).”

**Sawtooth Function**: The sawtooth function is the piecewise-linear function such that for each real number \( x \), the value of the function is given by the expression \( x - [x] \).

The sawtooth function assigns to each positive number the part of the number (the non-integer part) that is to the right of the floor of the number on the number line. That is, if we let \( f(x) = x - [x] \) for all real numbers \( x \) then

\[
\begin{align*}
  f\left(\frac{1}{3}\right) &= \frac{1}{3} f\left(\frac{1}{3}\right) = \frac{1}{3} f(1000.02) = 0.02, \\
  f(-0.3) &= 0.7, etc.
\end{align*}
\]

**Exit Ticket (5 minutes)**

Ensure that students complete the Exit Ticket independently, in order to provide evidence of each student’s understanding of the lesson.
Lesson 15: Piecewise Functions

Exit Ticket

Each graph shown below represents the solution set to a two-variable equation.

Graph A  
Graph B  
Graph C

1. Which of these graphs could be represented by a function? Explain your reasoning.

2. For each one that can be represented by a function, define a piecewise function whose graph would be identical to the solution set shown.
Exit Ticket Sample Solutions

1. Which of these graphs could be represented by a function? Explain your reasoning.
   *Graphs A and C could be represented by a function because each x in the domain is paired with exactly one y in the range.*

2. For each one that can be represented by a function, define a piecewise function whose graph would be identical to the solution set shown.
   
   **Graph A:**
   \[
   f(x) = \begin{cases} 
   -x - 1, & x < -1 \\
   x + 1, & x \geq -1
   \end{cases}
   \]

   **Graph C:**
   \[
   f(x) = \begin{cases} 
   -2, & x < 0 \\
   0, & x = 0 \\
   2, & x > 0
   \end{cases}
   \]

Problem Set Sample Solutions

These problems build student familiarity with piecewise functions and continue to reinforce the definition of function. The following solutions indicate an understanding of the objectives of this lesson:

1. Explain why the sawtooth function, \( \text{sawtooth}(x) = x - |x| \) for all real numbers \( x \), takes only the “fractional part” of a number when the number is positive.
   *If you subtract the integer part of a number from the number, only the “fractional part” will remain.*

2. Let \( g(x) = |x| - |x| \) where \( x \) can be any real number. In otherwords, \( g \) is the difference between the ceiling and floor functions. Express \( g \) as a piecewise function.
   \[
   g(x) = \begin{cases} 
   0 & x \text{ is an integer} \\
   1 & x \text{ is not an integer}
   \end{cases}
   \]

3. The Heaviside function is defined using the formula below.
   \[
   H(x) = \begin{cases} 
   -1, & x < 0 \\
   0, & x = 0 \\
   1, & x > 0
   \end{cases}
   \]

   Graph this function and state its domain and range.
   *Domain: All real numbers.*
   *Range: \((-1, 0, 1)\).*
4. The following piecewise function is an example of a step function.

\[ S(x) = \begin{cases} 
3 & -5 \leq x < -2 \\
1 & -2 \leq x < 3 \\
2 & 3 \leq x \leq 5 
\end{cases} \]

- a. Graph this function and state the domain and range.
  
  **Domain:** \([-5, 5]\), **Range:** \([1, 2, 3]\).

- b. Why is this type of function called a step function? The horizontal line segments step up and down like steps.

5. Let \( f(x) = \frac{|x|}{x} \) where \( x \) can be any real number except 0.

- a. Why is the number 0 excluded from the domain of \( f \)?
  
  If \( x = 0 \) then the expression would not be defined.

- b. What is the range of \( f \)?
  
  \((-1, 1)\).

- c. Create a graph of \( f \).

- d. Express \( f \) as a piecewise function.
  
  \[ f(x) = \begin{cases} 
-1 & x < 0 \\
1 & x > 0 
\end{cases} \]

- e. What is the difference between this function and the Heaviside function?
  
  The domain of the Heaviside function is all real numbers. The Heaviside function has a value of 0 when \( x = 0 \). This function excludes the real number 0 from the domain.
6. Graph the following piecewise functions for the specified domain.

a. \( f(x) = |x + 3| \) for \(-5 \leq x \leq 3\)

b. \( f(x) = |2x| \) for \(-3 \leq x \leq 3\)

c. \( f(x) = |2x - 5| \) for \(0 \leq x \leq 5\)
d. \( f(x) = |3x + 1| \) for \(-2 \leq x \leq 2\)

```
\[ f(x) = \begin{cases} 
3x + 1 & \text{if } -2 \leq x < -1 \\
1 & \text{if } -1 \leq x \leq 1 \\
3x - 1 & \text{if } 1 < x \leq 2
\end{cases} \]
```

e. \( f(x) = |x| + x \) for \(-5 \leq x \leq 3\)

```
\[ f(x) = \begin{cases} 
-x & \text{if } -5 \leq x < 0 \\
x & \text{if } 0 \leq x \leq 3
\end{cases} \]
```

f. \( f(x) = \begin{cases} 
x & \text{if } x \leq 0 \\
x + 1 & \text{if } x > 0
\end{cases} \)

```
\[ f(x) = \begin{cases} 
x & \text{if } x \leq 0 \\
x + 1 & \text{if } x > 0
\end{cases} \]
```
g. \[ f(x) = \begin{cases} \frac{2x + 3}{x} & \text{if } x < -1 \\ 3 - x & \text{if } x \geq -1 \end{cases} \]

7. Write a piecewise function for each graph below.

a. 

Answer:

\[ b(x) = \begin{cases} 2x + 4 & x \leq 0 \\ 4 & x > 0 \end{cases} \]
Lesson 15: Piecewise Functions

Date: 3/25/14

Lesson 15: Piecewise Functions

Graph of $p$:

$p(x) = \begin{cases} 
-3 & x < -2 \\
1 & -2 \leq x \leq 2 \\
4 & x > 2 
\end{cases}$

Graph of $k$:

$k(x) = \begin{cases} 
\frac{x + 3}{2} & x \leq -1 \\
\frac{x + 1}{2} & -1 \leq x \leq 1 \\
x + 1 & x \geq 1 
\end{cases}$

Graph of $h$:

$h(x) = \begin{cases} 
-4x - 4 & x < 0 \\
-2 & 0 \leq x \leq 2 \\
-2x + 8 & x > 2 
\end{cases}$

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Lesson 16: Graphs Can Solve Equations Too

Student Outcomes

- Students discover that the multi-step and exact way of solving \( |2x - 5| = |3x + 1| \) using algebra can sometimes be avoided by recognizing that an equation of the form \( f(x) = g(x) \) can be solved visually by looking for the intersection points of the graphs of \( y = f(x) \) and \( y = g(x) \).

Lesson Notes

This lesson focuses on A.REI.11 which emphasizes that the \( x \)-coordinates of the intersection points of the graphs of two functions \( f \) and \( g \) are the solutions to the equation \( f(x) = g(x) \). This lesson ties work from Module 1 on solving systems of two-variable equations to work with functions and leads students to the understanding of what the solution set to a one-variable equation can be.

Classwork

Opening Exercises 1–5 (5 minutes)

The first exercise is designed to use a practical application to motivate study of using the graphs of \( f \) and \( g \) to solve the associated equation, \( f(x) = g(x) \). Pause for discussion after students have had an opportunity to answer the question.

1. A cell phone company offers two different plans. The first plan costs $45 including 80 free texts, but each text after the first 80 costs $0.25. The second plan costs $70, including unlimited texts. These plans can be modeled by functions \( f \) and \( g \), where \( x \) represents the number of texts sent and \( f(x) \) and \( g(x) \) represent the cost of each plan, respectively. Below are the graphs of \( f \) and \( g \).

![Graph of f and g](image)

**Scaffolding:**

- Challenge students to come up with expressions for \( f(x) \) and \( g(x) \).
- Students may benefit from having the scenario read aloud.

**The intersection point represents where \( f(x) \) and \( g(x) \) have the same value. This point, (180,70), tells us that when \( x = 180 \), both \( f(x) = 70 \) and \( g(x) = 70 \). Another way of saying this is that when \( x = 180 \), \( f(x) = g(x) \). In the context of this problem, the intersection point tells us that 180 is the number of texts for which both plans cost the same amount, $70.**

**Notes:**

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2. Solve for $x$ in the following equation: $|x + 2| = 0.5x + 1$

$$|x + 2| = 0.5x + 4$$

$$x + 2 = 0.5x + 4$$
$$0.5x = 2$$
$$x = 4$$

$$x + 2 = -(0.5x + 4)$$
$$1.5x = -6$$
$$x = -4$$

3. Now let $f(x) = |x + 2| - 3$ and $g(x) = 0.5x + 1$. When does $f(x) = g(x)$? To answer this, first graph $y = f(x)$ and $y = g(x)$ on the same set of axes.

4. When does $f(x) = g(x)$? What is the visual significance of the points where $f(x) = g(x)$?

$f(x) = g(x)$ when $x = 4$ and when $x = -4$; $(4, 3)$ and $(-4, -1)$. The points where $f(x) = g(x)$ are the intersections of the graphs of $f$ and $g$.

5. Is each intersection point $(x, y)$ an element of the graph $f$ and an element of the graph of $g$? In other words, do the functions $f$ and $g$ really have the same value when $x = 4$? What about when $x = -4$?

Yes. You can determine this by substituting $x = -4$ and $x = 4$ into both $f$ and $g$.

$$-4 + 2 = 3$$
$$-4 + 2 = 3$$
$$-1 = 0.5(-4) + 1$$
$$3 = |4 + 2| - 3$$
$$3 = 0.5(4) + 1$$
$$-1 = 2 - 3$$
$$-1 = -1$$
$$3 = 6 - 3$$
$$3 = 3$$

Scaffolding:
- Use a simpler first example, such as $3 = |x - 2|$.
- Provide graphing calculators or software to aid in creating the graphs.
- As an extension, ask if it could ever be the case that $f(x)$ never equals $g(x)$, for two functions $f$ and $g$. Ask students to explain using examples.
Be sure to review the solutions to these problems with the entire class before moving on; use this as an opportunity to informally assess student understanding of the significance of the intersection of the graphs. Before sharing as a whole group, give students time to compare their answers with a partner.

**Discussion (8 minutes)**

Lead a discussion that ties together the work in the Opening Exercises. The idea here is to create an equation \( f(x) = g(x) \) and show that the \( x \)-coordinates of the intersection points are the solution set to this equation. This example will draw on MP.7 as students will need to look closely to determine the connection between the functions and equations involved.

First summarize what we know from the Opening Exercise. Ask students to discuss this with a partner and then call on a few people to share their thoughts. Make sure the following point is clear to the class:

- For functions \( f \) and \( g \), we have found special ordered pairs that (1) are points in the intersection of the graph of \( f \) and the graph of \( g \), (2) are solutions to a system of equations given by the equations \( y = f(x) \) and \( y = g(x) \), and (3) have \( x \)-values that satisfy the equation \( f(x) = g(x) \).

Write this equation on the board. Ask students to use the solutions from the Opening Exercise to answer the following:

- Are the \( x \)-coordinates of the intersection points solutions to this equation?
  - Yes, when I substituted the \( x \)-coordinates into the equation I got a true number sentence.
- Are the \( y \)-coordinates of the intersection points solutions to this equation?
  - No, when I substituted the \( y \)-coordinates into the equation I got a false number sentence.
- Do you think there are any other solutions to this equation? How could you be sure?
  - I do not think so because each side of the equation is one of the functions shown in the graphs. The shape of the graph makes me think that there are no other intersection points. We could algebraically solve the equation to prove that these are the only solutions.

Give students (in groups of three or four) time to debate the next discussion question. Have different students share their thinking with the whole class.

- Is it always true that the \( x \)-coordinates of the intersection points of the graphs of two functions will be the solution set to the equation \( f(x) = g(x) \)?
  - Yes. To create the graphs of \( f \) and \( g \) we cycle through some of the domain values \( x \) and plot the pairs \((x, f(x))\) and \((x, g(x))\). The points that these two functions have in common will have \( x \)-values that satisfy the equation \( f(x) = g(x) \) because this equation asks us to find the domain elements \( x \) that make the range elements \( f(x) \) and \( g(x) \) equal.
- What is the advantage of solving an equation graphically by finding intersection points in this manner?
  - It can be helpful when the equations are complicated or impossible to solve algebraically. It will also be useful when estimating solutions is enough to solve a problem. The graphically estimated solutions might give insight into ways to solve the equation algebraically.
Example 1 (8 minutes)

This example provides an opportunity to model explicitly how to use graphs of functions to solve an equation. As you work with students, guide them to label the graphs similarly to what is shown in the solutions below. This will reinforce proper vocabulary. In this guided example, students will complete the graphs of the functions and then fill in the blanks as you discuss as a whole class. This example should reinforce the previous discussion.

Example 1

Solve this equation by graphing two functions on the same Cartesian plane: \(|0.5x| - 5 = -|x - 3| + 4\).

Let \(f(x) = |0.5x| - 5\) and let \(g(x) = -|x - 3| + 4\) where \(x\) can be any real number.

We are looking for values of \(x\) at which the functions \(f\) and \(g\) have the same output value.

Therefore, we set \(y = f(x)\) and \(y = g(x)\) so we can plot the graphs on the same coordinate plane:

From the graph, determine the intersection points.

\((-4, -3)\) and \((8, -1)\)

How could we use the intersection points to find the solution set to the equation \(|0.5x| - 5 = -|x - 3| + 4\)?

The solution set to the original equation is \((-4, 8)\). When \(x = -4\), both \(f(x)\) and \(g(x)\) are \(-3\). When \(x = 8\), both \(f(x)\) and \(g(x)\) are \(-1\). Thus, the expressions \(|0.5x| - 5\) and \(-|x - 3| + 4\) are equal when \(x\) is \(-4\) or is \(8\).

After working with the class to use their knowledge from the previous lesson to create these graphs, lead a discussion that emphasizes the following:

- We are looking for values of \(x\) where the values \(f(x)\) and \(g(x)\) are the same. In other words, we want to identify the points \((x, f(x))\) of the graph of \(f\) and the points \((x, g(x))\) of the graph of \(g\) that are the same. This will occur where the graphs of the two functions intersect.
- We must also convince ourselves that these are the only two solutions to this equation. Pose the question: How can we be certain that these two intersection points are the only two solutions to this equation? Give students time to discuss this with a partner or in a small group. Encourage them to reason from the graphs of the functions, rather than solving the equation.

Scaffolding:

- Provide graphing calculators to aid in creating the graphs.
- If students are struggling to answer, consider using these scaffolded questions:
  - “What are the values of \(f(x)\) and \(g(x)\) at the intersection points?”
  - “What does this mean about the \(x\)-values that make \(|0.5x| - 5\) and \(-|x - 3| + 4\) equal?”

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For all \( x < -4 \) or \( x > 8 \), the differences in the y-values of two functions are always greater than zero. To see this, note that \( f(x) - g(x) = |0.5x| - 5 - (-|x - 3| + 4) = |0.5x| + |x - 3| - 9 \). The last expression is greater than zero when \( |0.5x| + |x - 3| > 9 \), which is certainly true for \( x < -4 \) or \( x > 8 \) (by inspection—there is no need to solve this equation due to the graph). Hence, the only solutions can occur in the interval \(-4 \leq x \leq 8\), of which there are two.

If time permits, challenge students to experiment with sketching in the same Cartesian plane the graphs of two functions (each of which involve taking an absolute value) that intersect at zero points, exactly one point, exactly two points, or an infinite number of points.

**Example 2 (10 minutes)**

This example requires graphing calculators or other graphing software that is capable of finding the intersection points of two graphs. As you work through this example, discuss and model how to:

- Enter functions into the graphing tool, graph them in an appropriate viewing window to see the intersection points, and use the features of the graphing technology to determine the coordinates of the intersection points.
- Show the difference between ‘tracing’ to the intersection point and using any built-in functions that determine the intersection point.
- Have students estimate the solutions from the graph before using the built-in features.
- Have students verify that the \( x \)-coordinates of the intersecting points are solutions to the equations.
- Have students sketch the graphs and label the coordinates of the intersection points on their handouts.

**Example 2**

Solve this equation graphically: \(-|x - 3.5| + 4 = 0.25x - 1\)

a. Write the two functions represented by each side of the equation.

\[ f(x) = -|x - 3.5| + 4 \quad \text{and} \quad g(x) = 0.25x - 1, \text{where } x \text{ can be any real number.} \]

b. Graph the functions in an appropriate viewing window.
c. Determine the intersection points of the two functions.

\((-2, -1.5) \text{ and } (6.8, 0.7)\)

d. Verify that the x-coordinates of the intersection points are solutions to the equation.

\[ \text{Let } x = -2 \text{ then} \]
\[ -|2 - 3.5| + 4 = 0.25(-2) - 1 \]
\[ -5.5 + 4 = -0.5 - 1 \]
\[ -1.5 = -1.5 \]

\[ \text{Let } x = 6.8 \text{ then} \]
\[ -|6.8 - 3.5| + 4 = 0.25(6.8) - 1 \]
\[ -3.3 + 4 = 1.7 - 1 \]
\[ 0.7 = 0.7 \]

Exercises 1–5 (8 minutes)

Before moving on to the exercises, consider asking students to summarize the lesson so far to their neighbor. Listen to the conversations to assess progress; take time to review the introductory example as necessary. When students are ready, they then practice using graphs of functions to solve equations. Students should work through these exercises independently or with a partner. Circulate among groups assessing informally and providing assistance as needed.

Exercises 1–5

Use graphs to find approximate values of the solution set for each equation. Use technology to support your work. Explain how each of your solutions relates to the graph. Check your solutions using the equation.

1. \[ 3 - 2x = |x - 5| \]

\( x = -2, \text{ the intersection point is } (-2, 7). \)

2. \[ 2(1.5)^x = 2 + 1.5x \]

\( \text{First solution is } x = 0, \text{ from the point } (0, 2); \)

\( \text{Second solution answers will vary, } x \text{ is about 2.7 or 2.8,} \)

\( \text{based on the actual intersection point of } (2.776, 6.164). \)

3. The graphs of the functions \( f \) and \( g \) are shown.

a. Use the graph to approximate the solution(s) to the equation \( f(x) = g(x) \).

\( \text{Based on the graphs, the approximate solutions are } (-0.7, 2). \)

b. Let \( f(x) = x^2 \) and let \( g(x) = 2^x \). Find all solutions to the equation \( f(x) = g(x) \). Verify any exact solutions that you determine using the definitions of \( f \) and \( g \). Explain how you arrived at your solutions.
By guessing and checking, \( x = 4 \) is also a solution of the equation because \( f(4) = 16 \) and \( g(4) = 16 \). Since the graph of the exponential function is increasing and increases more rapidly than the squaring function, there will only be 3 solutions to this equation. The exact solutions are \( x = 2 \) and \( x = 4 \) and an approximate solution is \( x = -0.7 \).

4. The graphs of \( f \), a function that involves taking an absolute value, and \( g \), a linear function, are shown to the right. Both functions are defined over all real values for \( x \). Tami concluded that the equation \( f(x) = g(x) \) has no solution.

Do you agree or disagree? Explain your reasoning.
I disagree with Tami because we cannot see enough of this graph. The graph of the function shown to the left has a slope of 5. The graph of the function shown to the right has a slope greater than 5. Therefore, these two functions will intersect somewhere in the first quadrant. We would have to ‘zoom out’ to see the intersection point.

5. The graphs of \( f \), a function that involves taking the absolute value, and \( g \), an exponential function, are shown below. Sharon said the solution set to the equation \( f(x) = g(x) \) is exactly \((-7, 5)\).

Do you agree or disagree with Sharon? Explain your reasoning.
I disagree with Sharon. We could say that the solution set is approximately \((-7, 5)\) but without having the actual equations or formulas for the two functions, we cannot be sure the \( x \)-values of the intersection points are exactly \(-7 \) and \( 5 \).

Closing (2 minutes)

In the last two exercises, students reflect on the limitations of solving an equation graphically. Debrief these exercises as a whole class and encourage different groups to present their reasoning to the entire class. Clarify any misconceptions before moving on, and give students time to revise their work. In Exercise 4, it is clear that there is an intersection point that is not visible in the viewing window provided. In Exercise 5, the intersection points would need to be estimated. If we do not have the exact algebraic solutions of the equation, then we can only estimate the solution set using graphs.

Exit Ticket (5 minutes)

Ensure that students complete the Exit Ticket independently, in order to provide evidence of each student’s understanding of the lesson.
Lesson 16: Graphs Can Solve Equations Too

Exit Ticket

1. How do intersection points of the graphs of two functions \( f \) and \( g \) relate to the solution of an equation in the form \( f(x) = g(x) \)?

2. What are some benefits of solving equations graphically? What are some limitations?
Exit Ticket Sample Solutions

1. How do intersection points of the graphs of two functions \( f \) and \( g \) relate to the solution of an equation in the form \( f(x) = g(x) \)?

   *The \( x \)-coordinates of the intersection points of the graphs of two functions are the solutions of the equation.*

2. What are some benefits of solving equations graphically? What are some limitations?

   **Benefits:** Solving equations graphically can be helpful when you don’t know how to solve the equation algebraically. It can also save you some time if you have technology available. This method can only provide approximate solutions, which may be all you need. Or the approximate solutions may give you insight into how to solve the equation algebraically.

   **Limitations:** You cannot be sure you have found all the solutions to an equation unless you can reason about the graphs of the functions themselves and convince yourself that no other intersection points are possible. The solutions found graphically rely on eyeballing. There is no guarantee that they are exact solutions; sometimes they are, but other times they are just decent approximations.

Problem Set Sample Solutions

1. Solve the following equations graphically. Verify the solution sets using the original equations.

   a. \( 2x - 4 = \sqrt{x + 5} \)

      *Approximately 3.4538*

   b. \( |x| = x^2 \)

      \((-1, 0, 1)\)

   c. \( x + 2 = x^3 - 2x - 4 \)

      *Approximately 2.3553*

   d. \( |3x - 4| = 5 - |x - 2| \)

      \(0.25, 2.75\)

   e. \( 0.5x^3 - 4 = 3x + 1 \)

      *Approximately 3.0467*

   f. \( 6 \left(\frac{1}{2}\right)^{5x} = 10 - 6x \)

      *Approximately –0.1765 and 1.6636*
In each problem, the graphs of the functions \( f \) and \( g \) are shown on the same Cartesian plane. Estimate the solution set to the equation \( f(x) = g(x) \). Assume that the graphs of the two functions only intersect at the points shown on the graph.

2. \([3, 9]\)

3. \((-3, 1)\)

4. \([1, 2]\)

5. \((1.2, 6)\)
6. The graph below shows Glenn’s distance from home as he rode his bicycle to school, which is just down his street. His next-door neighbor Pablo, who lives 100 m closer to the school, leaves his house at the same time as Glenn. He walks at a constant velocity, and they both arrive at school at the same time.

![Graph of Glenn and Pablo's distances](image)

a. Graph a linear function that represents Pablo’s distance from Glenn’s home as a function of time.

b. Estimate when the two boys pass each other.

They cross paths at about 2 minutes and 5 minutes. I can tell this by finding the x-coordinates of the intersection points of the graphs of the functions.

c. Write piecewise-linear functions to represent each boy’s distance and use them to verify your answer to part (b).

\[ P(t) = 100 + 37.5t \]

\[ G(t) = \begin{cases} 
\frac{250}{3}t & 0 \leq t \leq 3 \\
200 + \frac{50}{3}t & 3 < t \leq 6 \\
50t & 6 < t \leq 8 
\end{cases} \]

At about 2 minutes: \(100 + \frac{75}{2}t = \frac{250}{3}t\) or \(600 + 225t = 500t\) or \(275t = 600\) or \(t = \frac{24}{11}\) min.

At about 5 minutes: \(100 + \frac{75}{2}t = 200 + \frac{50}{3}t\) or \(225t = 600 + 100t\) or \(125t = 600\) or \(t = \frac{24}{5} = 4.8\) min.
Lesson 17: Four Interesting Transformations of Functions

Student Outcomes
- Students examine that a vertical translation of the graph of \( y = f(x) \) corresponds to changing the equation from \( y = f(x) \) to \( y = f(x) + k \).
- Students examine that a vertical scaling of the graph of \( y = f(x) \) corresponds to changing the equation from \( y = f(x) \) to \( y = kf(x) \).

Lesson Notes
Students enter Algebra 1 having experience with transforming lines, rays, triangles, etc., using translations, rotations, reflections, and dilations from Grade 8 Modules 2 and 3. Thus, it is natural to begin a discussion of transformations of functions by transforming graphs of functions—the graph of a function, \( f: \mathbb{R} \to \mathbb{R} \), is just another geometric figure in the (Cartesian) plane. Students use language such as, “a translation 2 units to the left,” or, “a vertical stretch by a scale factor of 3,” to describe how the original graph of the function is transformed into the new graph geometrically.

As students apply their Grade 8 geometry skills to the graph of the equation \( y = f(x) \), they realize that the translation of the graph to the right by \( 4 \) units is given by the graph of the equation \( y = f(x - 4) \). This recognition, in turn, leads to the idea of a transformation of a function. (i.e., a new function such that the graph of it is the transformation of the original graph of \( y = f(x) \).) In the example described, it is the function given by \( g(x) = f(x - 4) \) for any real number \( x \) such that \( x - 4 \) is in the domain of \( f \).

Since the transformation of the function is itself another function (and not a graph), we must use function language to describe the transformation. A function \( f \) cannot be translated up, down, right, or left (even though its graph can). Rather, students can use function language such as: “For the same inputs, the values of the transformed function are two times as large as the values of the original function.”

Focusing on standard F-BF.B.3, these lessons encourage fluidity in both the language associated with transformations of graphs and the language associated with transformations of functions. While a formal definition for the transformation of a function is not included, teachers are encouraged use language precisely as students work to develop the notion of transformation of a function and relate it to their understanding of transformations of graphical objects. As the focus of the lesson is on the effect of the constant, \( k \), the use of graphing calculators or graphing software is strongly recommended, so that students may focus on this concept rather than the process of plotting points.

In the exploratory challenge, you may highlight MP.3 by asking students to make a conjecture about the effect of \( k \). Use this as an opportunity to build students’ abilities to express their reasoning by writing and speaking; ensure also that students have opportunities to critique the reasoning of others. Consider sharing a hypothetical student explanation or conjecture and critiquing as a class. This challenge also calls on students to employ MP.8, as they will generalize the effect of \( k \) through repeated graphing. Ensure ample opportunity for students to create graphs repeatedly and describe fully the effect of \( k \).
Classwork

Exploratory Challenge 1/Example 1 (12 minutes)

Let \( f(x) = |x| \) for all real numbers \( x \). Students explore the effect on the graph of \( y = f(x) \) by changing the equation \( y = f(x) \) to \( y = f(x) + k \) for given values of \( k \). Allow students to work independently or with a partner.

\[
\text{Exploratory Challenge 1/Example 1}
\]

Let \( f(x) = |x|, g(x) = f(x) - 3, h(x) = f(x) + 2 \) for any real number \( x \).

1. Write an explicit formula for \( g(x) \) in terms of \( |x| \) (i.e., without using \( f(x) \) notation):
   \[ g(x) = |x| - 3 \]

2. Write an explicit formula for \( h(x) \) in terms of \( |x| \) (i.e., without using \( f(x) \) notation):
   \[ h(x) = |x| + 2 \]

3. Complete the table of values for these functions.

| \( x \) | \( f(x) = |x| \) | \( g(x) = f(x) - 3 \) | \( h(x) = f(x) + 2 \) |
|-------|----------------|----------------|----------------|
| -3    | 3              | 0              | 5              |
| -2    | 2              | -1             | 4              |
| -1    | 1              | -2             | 3              |
| 0     | 0              | -3             | 2              |
| 1     | 1              | -2             | 3              |
| 2     | 2              | -1             | 4              |
| 3     | 3              | 0              | 5              |

4. Graph all three equations: \( y = f(x), y = f(x) - 3, \) and \( y = f(x) + 2 \).

**Scaffolding:**
- Give students access to graphing calculators or computer graphing utilities to lessen the procedural burden and allow them to focus on the effect of \( k \) on the graph.
- For students in need of a challenge, consider asking them to graph more technically challenging examples, such as \( m(x) = f(x) + \frac{3}{4} \) or to develop a conjecture about the appearance of the graph of an extreme case, such as \( p(x) = f(x) - 1000 \).
5. What is the relationship between the graph of \( y = f(x) \) and the graph of \( y = f(x) + k \)?

For values of \( k \), where \( k > 0 \), for every point \( (x, f(x)) \) that satisfies the equation \( y = f(x) \), there is a corresponding point \( (x, f(x) + k) \) on the graph, located \( k \) units above \( (x, f(x)) \) that satisfies the equation \( y = f(x) + k \). The graph of \( y = f(x) + k \) is the vertical translation of the graph of \( y = f(x) \) by \( k \) units upward.

For values of \( k \), where \( k < 0 \), for every point \( (x, f(x)) \) that satisfies the equation \( y = f(x) \), there is a corresponding point \( (x, f(x) + k) \) on the graph, located \( k \) units below \( (x, f(x)) \) that satisfies the equation \( y = f(x) + k \). The graph of \( y = f(x) + k \) is the vertical translation of the graph of \( y = f(x) \) by \( k \) units downward.

The use of transformation language like “vertical translation” is purposeful. The Common Core State Standards require students to spend a lot of time talking about translations, rotations, reflections, and dilations in Grade 8. They will spend more time in Grade 10. Reinforcing this vocabulary will help to link these grades together.

6. How do the values of \( g \) and \( h \) relate to the values of \( f \)?

For each \( x \) in the domain of \( f \) and \( g \), the value of \( g(x) \) is 3 less than the value of \( f(x) \). For each \( x \) in the domain of \( f \) and \( h \), the value of \( h(x) \) is 2 more than the value of \( f(x) \).

Discussion (3 minutes)

Students should finish Example 1 with the understanding that the graph of a function \( g \) found by adding a number to another function, as in \( g(x) = f(x) + k \), is the translation of the graph of the function \( f \) vertically by \( k \) units (positively or negatively depending on the sign of \( k \)). Observe student responses to question 6 to assess understanding of the effect of \( k \), then discuss the activity, explicitly instructing as necessary.

Exploratory Challenge 2/Example 2 (12 minutes)

Let \( f(x) = |x| \) for any real number \( x \). Students explore the effect on the graph of \( y = f(x) \) by changing the equation \( y = f(x) \) to \( y = kf(x) \) for given values of \( k \). To continue to highlight MP.3, consider asking students to make a conjecture about the effect of \( k \) in this case, either in writing or with their partner. Allow opportunities for students to respond to the conjectures of others.

### Exploratory Challenge 2/Example 2

1. Let \( f(x) = |x|, g(x) = 2f(x), h(x) = \frac{1}{2}f(x) \) for any real number \( x \). Write a formula for \( g(x) \) in terms of \( |x| \) (i.e., without using \( f(x) \) notation):

   \[ g(x) = 2|x| \]

2. Write a formula for \( h(x) \) in terms of \( |x| \) (i.e., without using \( f(x) \) notation):

   \[ h(x) = \frac{1}{2}|x| \]
3. Complete the table of values for these functions.

| $x$ | $f(x) = |x|$ | $g(x) = 2f(x)$ | $h(x) = \frac{1}{2}f(x)$ |
|-----|-------------|---------------|-----------------|
| $-3$ | 3 | 6 | 1.5 |
| $-2$ | 2 | 4 | 1 |
| $-1$ | 1 | 2 | 0.5 |
| 0 | 0 | 0 | 0 |
| 1 | 1 | 2 | 0.5 |
| 2 | 2 | 4 | 1 |
| 3 | 3 | 6 | 1.5 |

4. Graph all three equations: $y = f(x), y = 2f(x)$, and $y = \frac{1}{2}f(x)$.

Scaffolding:
- Similar to before, provide graphing calculators or utilities to lessen the procedural burden and allow them to focus on the effect of $k$ on the graph.
- For students in need of a challenge, ask them to graph more technically challenging examples, such as $m(x) = 0.15f(x)$ or to develop a conjecture about the appearance of the graph of an extreme case, such as $p(x) = 1000f(x)$.

Given $f(x) = |x|$, let $p(x) = -|x|$, $q(x) = -2f(x)$, $r(x) = -\frac{1}{2}f(x)$ for any real number $x$.

5. Write the formula for $q(x)$ in terms of $|x|$ (i.e., without using $f(x)$ notation):

$$q(x) = -2|x|$$
6. Write the formula for \( r(x) \) in terms of \(|x|\) (i.e., without using \( f(x) \) notation):

\[ r(x) = -\frac{1}{2}|x| \]

7. Complete the table of values for the functions \( p(x) = -|x|, q(x) = -2f(x), r(x) = -\frac{1}{2}f(x) \).

| \( x \) | \( p(x) = -|x| \) | \( q(x) = -2f(x) \) | \( r(x) = -\frac{1}{2}f(x) \) |
|-------|----------------|--------------------|--------------------|
| -3    | -3             | -6                 | -1.5               |
| -2    | -2             | -4                 | -1                 |
| -1    | -1             | -2                 | -0.5               |
| 0     | 0              | 0                  | 0                  |
| 1     | -1             | -2                 | -0.5               |
| 2     | -2             | -4                 | -1                 |
| 3     | -3             | -6                 | -1.5               |

8. Graph all three functions on the same graph that was created in Problem 4. Label the graphs as \( y = p(x) \), \( y = q(x) \), and \( y = r(x) \).
9. How is the graph of \( y = f(x) \) related to the graph of \( y = kf(x) \) when \( k > 1 \)?

The graph of \( y = kf(x) \) for \( k > 1 \) contains points \((x, ky)\) which are related to points \((x, y)\) in the graph of \( y = f(x) \). The number \( ky \) is a multiple of \( y \); each \( y \)-value of \( y = g(x) \) is \( k \) times the \( y \)-value of \( y = f(x) \). The graph of \( y = kf(x) \) is a vertical scaling that appears to stretch the graph of \( y = f(x) \) vertically by a factor of \( k \).

10. How is the graph of \( y = f(x) \) related to the graph of \( y = kf(x) \) when \( 0 < k < 1 \)?

The graph of \( y = kf(x) \) for \( 0 < k < 1 \) contains points \((x, ky)\) which are related to points \((x, y)\) in the graph of \( y = f(x) \). The number \( ky \) is a fraction of \( y \); each \( y \)-value of \( y = g(x) \) is \( k \) times the \( y \)-value of \( y = f(x) \). The graph of \( y = kf(x) \) is a vertical scaling that appears to shrink the graph of \( y = f(x) \) vertically by a factor of \( k \).

11. How do the values of functions \( p, q, \) and \( r \) relate to the values of functions \( f, g, \) and \( h \), respectively? What transformation of the graphs of \( f, g, \) and \( h \) represents this relationship?

Each function is the opposite of the corresponding function. The result is that each \( y \)-value of any point on the graph of \( y = p(x), y = q(x), \) and \( y = r(x) \) are the opposite of the \( y \)-value of the graphs of the equations \( y = f(x), y = g(x), \) and \( y = h(x) \). Each graph is a reflection of the corresponding graph over the \( x \)-axis.

Discussion (3 minutes)

Students should finish Example 2 with the understanding that a number, a scale factor, multiplied to a function vertically scales the original graph. Consider asking students to summarize what they learned to their neighbor; use this as an opportunity to informally assess understanding before moving on. For a vertical scale factor of \( k > 1 \), the graph is a vertical stretch of the original graph; for a vertical scale factor of \( k \) where \( 0 < k < 1 \), the graph is a vertical shrink of the original graph. For a vertical scale factor of \( k \) where \( -1 < k < 0 \), the graph of the function is a reflection across the \( x \)-axis of the graph when \( 0 < k < 1 \). Similarly, for a vertical scale factor of \( k < -1 \), the graph is the reflection across the \( x \)-axis of the graph when \( k > 1 \).

Exercises (8 minutes)

Students complete exercises independently; then compare/discuss with partner or small group. Circulate to ensure that students grasp the effects of the given transformations.

Exercises

1. Make up your own function \( f \) by drawing the graph of it on the Cartesian plane below. Label it as the graph of the equation, \( y = f(x) \). If \( b(x) = f(x) - 4 \) and \( c(x) = \frac{1}{4} f(x) \) for every real number \( x \), graph the equations \( y = b(x) \) and \( y = c(x) \) on the same Cartesian plane.

Answers will vary. Look for and encourage students to create interesting graphs for their function \( f \). (Functions DO NOT have to be defined by algebraic expressions—any graph that satisfies the definition of a function will do.) One such option is using \( f(x) = |x| \), as shown in the example below.

![Graph of Functions](image-url)
If time permits, have students present their graphs to the class and explain how they found the graphs of \( y = b(x) \) and \( y = c(x) \). Pay close attention to how students explain how they found the graph of \( y = c(x) \). Many might actually describe a horizontal scaling (or some other transformation that takes each point \((x, y)\) of the graph to another point that does not have the same \(x\)-coordinate). Stress that multiplying the function \( f \) by \( k \) only scales the \(y\)-coordinate and leaves the \(x\)-coordinate alone.

### Closing (3 minutes)

Ask students to summarize the ideas developed in this lesson, either in writing or with a neighbor. Use this as an opportunity to informally assess students’ understanding of the key ideas. Consider also these discussion ideas for the whole class:

- Discuss how the graph of \( y = f(x) \) can be vertically translated by positive or negative \( k \). Draw a graph of a made up function on the board, labeled by \( y = f(x) \), and show how to translate it up or down by \( k \) using the equation \( y = f(x) + k \).
- Discuss how the graph of \( y = f(x) \) can be vertically scaled by \( k \) for \( 0 < k < 1, \ k > 1, -1 < k < 0, \ k < -1 \). Use the graph of \( y = f(x) \) to show how to vertically scale (i.e., vertically stretch or shrink) by \( k \) units using the equation \( y = kf(x) \).

### Exit Ticket (5 minutes)

Ensure that students complete the Exit Ticket independently, in order to provide evidence of each student’s understanding of the lesson.
Lesson 17: Four Interesting Transformations of Functions

Exit Ticket

Let \( p(x) = |x| \) for every real number \( x \). The graph of \( y = p(x) \) is shown below.

1. Let \( q(x) = -\frac{1}{2}|x| \) for every real number \( x \). Describe how to obtain the graph of \( y = q(x) \) from the graph of \( y = p(x) \). Sketch the graph of \( y = q(x) \) on the same set of axes as the graph of \( y = p(x) \).

2. Let \( r(x) = |x| - 1 \) for every real number \( x \). Describe how to obtain the graph of \( y = r(x) \) from the graph of \( y = p(x) \). Sketch the graph of \( y = r(x) \) on the same set of axes as the graphs of \( y = p(x) \) and \( y = q(x) \).
Exit Ticket Sample Solutions

Let \( p(x) = |x| \) for every real number \( x \). The graph of \( y = p(x) \) is shown below.

1. Let \( q(x) = -\frac{1}{2}|x| \) for every real number \( x \). Describe how to obtain the graph of \( y = q(x) \) from the graph of \( y = p(x) \). Sketch the graph of \( y = q(x) \) on the same set of axes as the graph of \( y = p(x) \).

   *Reflect and vertically scale the graph of \( y = p(x) \) by plotting \( (x, -\frac{1}{2}y) \) for each point \((x, y)\) in the graph of \( y = p(x) \). See the graph of \( q(x) \) below.*

2. Let \( r(x) = |x| - 1 \) for every real number \( x \). Describe how to obtain the graph of \( y = r(x) \) from the graph of \( y = p(x) \). Sketch the graph of \( y = r(x) \) on the same set of axes as the graphs of \( y = p(x) \) and \( y = q(x) \).

   *Translate the graph of \( y = p(x) \) vertically down 1 unit. See the graph of \( y = r(x) \) below.*

![Graph of y = p(x), y = q(x), and y = r(x)](image)

Problem Set Sample Solutions

Let \( f(x) = |x| \) for every real number \( x \). The graph of \( y = f(x) \) is shown below. Describe how the graph for each function below is a transformation of the graph of \( y = f(x) \). Then use this same set of axes to graph each function for problems 1 – 5. Be sure to label each function on your graph (by \( y = a(x), y = b(x) \), etc.).

1. \( a(x) = |x| + \frac{3}{2} \)

   *Translate the graph of \( y = f(x) \) up 1.5 units.*

2. \( b(x) = -|x| \)

   *Reflect \( y = f(x) \) across the x-axis.*
3. \( c(x) = 2|x| \)
   Vertically scale/stretch the graph of \( y = f(x) \) by doubling the output values for every input.

4. \( d(x) = \frac{1}{3}|x| \)
   Vertically scale/shrink the graph of \( y = f(x) \) by dividing the output values by 3 for every input.

5. \( e(x) = |x| - 3 \)
   Translate the graph of \( y = f(x) \) down 3 units.

6. Let \( r(x) = |x| \) and \( t(x) = -2|x| + 1 \) for every real number \( x \). The graph of \( y = r(x) \) is shown below. Complete the table below to generate output values for the function \( t \); then graph the equation \( y = t(x) \) on the same set of axes as the graph of \( y = r(x) \).

| \( x \) | \( r(x) = |x| \) | \( t(x) = -2|x| + 1 \) |
|-------|-----------------|-----------------|
| -2    | 2               | -3              |
| -1    | 1               | -1              |
| 0     | 0               | 1               |
| 1     | 1               | -1              |
| 2     | 2               | -3              |
7. Let \( f(x) = |x| \) for every real number \( x \). Let \( m \) and \( n \) be functions found by transforming the graph of \( y = f(x) \). Use the graphs of \( y = f(x), y = m(x), \) and \( y = n(x) \) below to write the functions \( m \) and \( n \) in terms of the function \( f \). (Hint: what is the \( k \)?)

\[
\begin{align*}
m(x) &= 2f(x) \\
n(x) &= f(x) + 2
\end{align*}
\]
Lesson 18: Four Interesting Transformations of Functions

Student Outcomes

- Students examine that a horizontal translation of the graph of \( y = f(x) \) corresponds to changing the equation from \( y = f(x) \) to \( y = f(x - k) \).

Lesson Notes

In Lesson 18, focus continues on standard F-BF.B.3. Students examine horizontal translations (shifts) in the graph of a function and how they are represented in the equation of the function. Students will contrast the horizontal shift to the vertical shift covered in Lesson 17. They should be able to describe the transformations of the graph associated with the transformation of the function, as well as write the equation of a graph based on the translations (shifts) or vertical scalings (stretches) of another graph whose equation is known.

In a way very similar to Lesson 17, students will engage deeply with MP.3 and MP.8 as they make conjectures about the effect of \( k \) and then engage in repeated graphing examples to make a generalization that either confirms or refutes their conjectures. A common misconception that students may have is confusing the “direction” of the translation, assuming that \( f(x + k) \) will result in a transformation by \( k \) units to the right (for \( k > 0 \)). This is a misunderstanding to watch for and highlight in discussion.

Classwork

Example 1 (8 minutes)

Students explore that a horizontal translation of the graph of \( y = f(x) \) corresponds to changing the equation from \( y = f(x) \) to \( y = f(x - k) \) for given values of \( k \). As an example of MP.3, ask students to make a conjecture about how they believe this placement of \( k \) will affect the graph, either in writing or with a partner.

Example 1

Let \( f(x) = |x|, g(x) = f(x - 3), h(x) = f(x + 2) \) where \( x \) can be any real number.

a. Write the formula for \( g(x) \) in terms of \( |x| \) (i.e., without using \( f(x) \) notation):
   \[ g(x) = |x - 3| \]

b. Write the formula for \( h(x) \) in terms of \( |x| \) (i.e., without using \( f(x) \) notation):
   \[ h(x) = |x + 2| \]

Scaffolding:

- Provide graphing calculators or utilities in order to allow all students to focus on the concept being developed in this example.
- For students that are ready for an extension, consider asking them to make a conjecture about an extreme case, such as \( k(x) = f(x + 1000) \) or a case that combines concepts from Lesson 17, such as \( m(x) = f(x - 3) + 4 \).
c. Complete the table of values for these functions.

|   | \( f(x) = |x| \) | \( g(x) = f(x - 3) \) | \( h(x) = f(x + 2) \) |
|---|---|---|---|
| -3 | 3 | 6 | 1 |
| -2 | 2 | 5 | 0 |
| -1 | 1 | 4 | 1 |
| 0  | 0 | 3 | 2 |
| 1  | 1 | 2 | 3 |
| 2  | 2 | 1 | 4 |
| 3  | 3 | 0 | 5 |

d. Graph all three equations: \( y = f(x) \), \( y = f(x - 3) \), and \( y = f(x + 2) \).

![Graph of functions](image)

e. How does the graph of \( y = f(x) \) relate to the graph of \( y = f(x - 3) \)?

*The graph of \( f(x - 3) \) is the graph of \( f(x) \) translated horizontally to the right 3 units.*

f. How does the graph of \( y = f(x) \) relate to the graph of \( y = f(x + 2) \)?

*The graph of \( f(x + 2) \) is the graph of \( f(x) \) translated horizontally to the left 2 units.*

g. How does the graph of \( y = |x| - 3 \) and the graph of \( y = |x - 3| \) relate differently to the graph of \( y = |x| \)?

*The graph of \( y = |x| - 3 \) translates the graph of \( y = |x| \) down 3 units whereas the graph of \( y = |x - 3| \) translates the graph of \( y = |x| \) to the right 3 units.*

h. How do the values of \( g \) and \( h \) relate to the values of \( f \)?

*The input value for \( g \) has to be 3 more than the input value for \( f \) to get the same output values. The input value for \( h \) has to be two less than the input value for \( f \) to get the same output values.*
Discussion (5 minutes)

Students should finish Example 1 with the understanding that the graph of a function \( g \) found by subtracting a number \( k \) to the input of another function, as in \( g(x) = f(x - k) \), is a translation of the graph of the function \( f \) horizontally by \( k \) units (positively or negatively, depending on the sign of \( k \)).

- If we replace 3 by a number \( k \) in \( g(x) = f(x - 3) \) as in Example 1 to get \( g(x) = f(x - k) \), explain how to translate the graph of \( f \) to the graph of \( g \) in terms of \( k \).
  - If \( k > 0 \), then the graph of \( f \) is translated to the right by \( |k| \) units.
  - If \( k < 0 \), then the graph of \( f \) is translated to the left by \( |k| \) units.
  - In general, for any \( k \), the graph of \( f \) is translated horizontally by \( k \) units (where \( k > 0 \) corresponds to a translation to the right and \( k < 0 \) corresponds to a translation to the left).

- How does your answer for \( k < 0 \) make sense for \( h(x) = f(x + 2) \)?
  - We can rewrite \( h(x) = f(x + 2) \) as \( h(x) = f(x - (-2)) \). Therefore, since \( -2 < 0 \), the graph of \( h \) should be the translation of the graph of \( f \) to the left by \( |-2| \) units.

- What concept from Grade 8 geometry best describes the shifts of the graphs of the functions in Example 1?
  - Translation. In fact, we use the word “translate” to help you remember.

- Students should be comfortable explaining the difference between the translations of the graphs \( y = |x| + k \) and \( y = |x + k| \).

- Students may confuse the direction of a horizontal translation since the equation may seem to indicate the “opposite” direction (i.e., \( y = |x + 3| \) may be confused as a translation to the right because of the addition of 3 to \( x \)), especially since a vertical translation up is the transformation given by adding a positive number \( k \) to the function. Help students articulate why the horizontal translation behaves as it does.

- Consider the function \( g(x) = |x - 3| \) and its graph from Example 1. There is a point \((x + 3, g(x + 3))\) on the graph of \( f \). We have \( g(x + 3) = f(x + 3 - 3) = f(x) \). Then the point \((x + 3, f(x))\) is on the graph of \( g \). Since \((x, f(x))\) is on the graph of \( f \) and \((x + 3, f(x))\) is \((x, f(x))\) shifted 3 units to the right, we conclude that the graph of \( g \) is the graph of \( f \) translated 3 units to the right. A similar argument can be made for the graph of \( h \).

Exercises 1–3 (15 minutes)

Have students discuss the following three exercises in pairs; consider reading the first exercise out loud and asking students to summarize the argument first in their own words before responding. Discuss the answers as a class. This is another opportunity for students to engage in MP.3 as they respond to a hypothetical student’s argument.

**Exercises 1–3**

1. Karla and Isamar are disagreeing over which way the graph of the function \( g(x) = |x + 3| \) is translated relative to the graph of \( f(x) = |x| \). Karla believes the graph of \( g \) is “to the right” of the graph of \( f \); Isamar believes the graph is “to the left.” Who is correct? Use the coordinates of the vertex of \( f \) and \( g \) to support your explanation.

   The graph of \( g \) is the graph of \( f \) translated to the left. The vertex of the graph of \( f \) is the point \((0, 0)\), whereas the vertex of the graph of \( g \) is the point \((-3, 0)\).

Note that in this lesson, students are working with translations of the function \( f(x) = |x| \). This function was chosen...
because it is one of the easier functions to use in showing how translations behave—just follow what happens to the vertex. We know that (0,0), or the vertex, is the point of the graph of \( f \) where the function’s outputs change between decreasing and increasing. As a horizontal translation, the vertex of the graph of \( g \) will also have a \( y \)-coordinate of 0; in fact, the vertex is \((-3,0)\). Thus the graph of \( f \) is translated 3 units to the left to get the graph of \( g \).

2. Let \( f(x) = |x| \) where \( x \) can be any real number. Write a formula for the function whose graph is the transformation of the graph of \( f \) given by the instructions below.
   a. A translation right 5 units.
      \[ a(x) = |x - 5| \]
   b. A translation down 3 units.
      \[ b(x) = |x| - 3 \]
   c. A vertical scaling (a vertical stretch) with scale factor of 5.
      \[ c(x) = 5|x| \]
   d. A translation left 4 units.
      \[ d(x) = |x + 4| \]
   e. A vertical scaling (a vertical shrink) with scale factor of \( \frac{1}{3} \).
      \[ e(x) = \frac{1}{3}|x| \]

3. Write the formula for the function depicted by the graph.
   a. \( y = |x + 6| \)

*Scaffolding:*
- Provide graphing calculators or utilities for students to use. This allows students to experiment with different values of \( k \) in developing their responses.
- As an extension, ask students to write equations from more complicated verbal descriptions, such as “Write an absolute value function whose graph has a vertex at \((-2,0)\) that includes the point \((-4,5)\).”
b. \( y = -2|x| \)

c. \( y = |x - \frac{3}{2}| \)
Exercises 4–5 (12 minutes)

Students now examine questions where more than one change is applied to \( f(x) = |x| \). Consider asking students to tackle these exercises independently.

**Exercises 4–5**

4. Let \( f(x) = |x| \) where \( x \) can be any real number. Write a formula for the function whose graph is the described transformation of the graph of \( f \).
   
a. A translation 2 units left and 4 units down.
   \[ y = |x + 2| - 4 \]

b. A translation 2.5 units right and 1 unit up.
   \[ y = |x - 2.5| + 1 \]
c. A vertical scaling with scale factor $\frac{1}{2}$ and then a translation 3 units right.

$$y = \frac{1}{2}|x - 3|$$

d. A translation 5 units right and a vertical scaling by reflecting across the x-axis with vertical scale factor $-2$.

$$y = -2|x - 5|$$

5. Write the formula for the function depicted by the graph.

a. $$y = |x + 2| - 4$$

b. $$y = |x - 5| - 2$$

c. $$y = -|x + 4|$$
Closing (2 minutes)

Ask students to respond to these questions, either in writing or with a partner. Use this as an opportunity to informally assess understanding of the ideas in this lesson.

- How can the graph of $y = f(x)$ be horizontally translated by positive or negative $k$?
- Draw a graph of a made-up function on the board, labeled by $y = f(x)$, and show how to translate it right or left by $k$ units using the equation $y = f(x - k)$.

Exit Ticket (3 minutes)

Ensure that students complete the Exit Ticket independently, in order to provide evidence of each student’s understanding of the lesson.
Lesson 18: Four Interesting Transformations of Functions

Exit Ticket

Write the formula for the functions depicted by the graphs below:

a. \( f(x) = \) ________________

b. \( g(x) = \) ________________

c. \( h(x) = \) ________________

\[
\begin{align*}
\text{Graph} & : y = f(x) \\
\text{Graph} & : y = g(x) \\
\text{Graph} & : y = h(x)
\end{align*}
\]
Exit Ticket Sample Solutions

Write the formula for the functions depicted by the graphs below:

a. \( f(x) = |x - 5| - 4 \)

b. \( g(x) = |x - 1| + 3 \)

c. \( h(x) = |x + 6| - 2 \)

Problem Set Sample Solutions

1. Working with quadratic functions.

a. The vertex of the quadratic function \( f(x) = x^2 \) is at \((0, 0)\), which is the minimum for the graph of \( f \). Based on your work in this lesson, to where do you predict the vertex will be translated for the graphs of \( g(x) = (x - 2)^2 \) and \( h(x) = (x + 3)^2 \)?

The vertex of \( g \) will be at \((2, 0)\); The vertex of \( h \) will be at \((-3, 0)\).

b. Complete the table of values and then graph all three functions.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = x^2 )</th>
<th>( g(x) = (x - 2)^2 )</th>
<th>( h(x) = (x + 3)^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>9</td>
<td>25</td>
<td>0</td>
</tr>
<tr>
<td>-2</td>
<td>4</td>
<td>16</td>
<td>1</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>9</td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>16</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>0</td>
<td>25</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>1</td>
<td>36</td>
</tr>
</tbody>
</table>

\( y = h(x) \)

\( y = f(x) \)

\( y = g(x) \)
2. Let $f(x) = |x - 4|$ for every real number $x$. The graph of the equation $y = f(x)$ is provided on the Cartesian plane below. Transformations of the graph of $y = f(x)$ are described below. After each description, write the equation for the transformed graph. Then, sketch the graph of the equation you write for part (d).

a. Translate the graph left 6 units and down 2 units.
   \[ y = |x + 2| - 2 \quad \text{or} \quad y = f(x + 2) - 2 \]

b. Reflect the resulting graph from part (a) across the $x$-axis.
   \[ y = -|x + 2| + 2 \quad \text{or} \quad y = -(f(x + 2) - 2) \]

c. Scale the resulting graph from part (b) vertically by a scale factor of $\frac{1}{2}$.
   \[ y = -\frac{1}{2}|x + 2| + 1 \quad \text{or} \quad y = -\frac{1}{2}(f(x + 2) - 2) \]

d. Translate the resulting graph from part (c) right 3 units and up 2 units. Graph the resulting equation.
   \[ y = -\frac{1}{2}|x - 1| + 3 \quad \text{or} \quad y = -\frac{1}{2}(f(x - 1) - 2) + 2 \]

3. Let $f(x) = |x|$ for all real numbers $x$. Write the formula for the function represented by the described transformation of the graph of $y = f(x)$.

a. First, a vertical stretch with scale factor $\frac{1}{3}$ is performed, then a translation right 3 units, and finally a translation down 1 unit.
   \[ a(x) = \frac{1}{3}|x - 3| - 1 \]

b. First, a vertical stretch with scale factor 3 is performed, then a reflection over the $x$-axis, then a translation left 4 units, and finally a translation up 5 units.
   \[ b(x) = -3|x + 4| + 5 \]
c. First, a reflection across the x-axis is performed, then a translation left 4 units, then a translation up 5 units, and finally a vertical stretch with scale factor 3.

\[ c(x) = -3|x + 4| + 15 \]

d. Compare your answers to parts (b) and (c). Why are they different?

*In part (c), the vertical stretch happens at the end, which means the graph resulting from the first three transformations is what is vertically stretched: \( c(x) = 3(-|x + 4| + 5) \).*

4. Write the formula for the function depicted by each graph.

a. \( a(x) = \frac{1}{2}|x - 1| - 3 \)

\[ \begin{array}{c}
-4 & -2 & 0 & 2 & 4 \\
-4 & -2 & 0 & 2 & 4 \\
\end{array} \]

\[ \begin{array}{c}
-4 & -2 & 0 & 2 & 4 \\
-4 & -2 & 0 & 2 & 4 \\
\end{array} \]

b. \( b(x) = -2|x + 3| + 4 \)

\[ \begin{array}{c}
-6 & -4 & -2 & 0 & 2 \\
-4 & -2 & 0 & 2 & 4 \\
\end{array} \]
Lesson 19: Four Interesting Transformations of Functions

Student Outcomes

- Students examine that a horizontal scaling with scale factor $k$ of the graph of $y = f(x)$ corresponds to changing the equation from $y = f(x)$ to $y = f\left(\frac{1}{k}x\right)$.

Lesson Notes

In this lesson, focus continues on F-BF.B.3 as students study the effect a horizontal scaling by scale factor $k$ has on the graph of an equation $y = f(x)$. For example, if $0 < k < 1$, a horizontal scaling by $k$ will horizontally shrink any geometric figure in the Cartesian plane, including figures that are graphs of functions. The horizontal scaling of a graph corresponds to changing the equation from $y = f(x)$ to $y = f\left(\frac{1}{k}x\right)$. For values of scale factor $k$ where $k > 1$, the graph of $y = f\left(\frac{1}{k}x\right)$ is a horizontal stretch of the graph of $y = f(x)$ by a factor of $k$.

Similar to lessons 17 and 18, in this lesson, students may employ MP.3 when they make conjectures about the effect of $k$, MP.8 when they use repeated reasoning to determine the effect of $k$, and MP.6 when they communicate the effect to others using careful language.

Classwork

Students explore the horizontal scaling of the graph of $y = f(x)$ when the equation changes from $y = f(x)$ to $y = f\left(\frac{1}{k}x\right)$ for $0 < k < 1$. In this case, students see the graph of $f$ is a horizontal “shrink” by $k$. In Example 1, the scale factor for $g$ is $k = \frac{1}{2}$, $g(x) = f\left(\frac{1}{2}x\right)$, or $g(x) = f(2x)$.

Example 1 (8 minutes)

Example 1

Let $f(x) = x^2$ and $g(x) = f(2x)$, where $x$ can be any real number.

a. Write the formula for $g$ in terms of $x^2$ (i.e., without using $f(x)$ notation):

$$g(x) = (2x)^2$$
b. Complete the table of values for these functions.

<table>
<thead>
<tr>
<th>x</th>
<th>f(x) = x^2</th>
<th>g(x) = f(2x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>9</td>
<td>36</td>
</tr>
<tr>
<td>-2</td>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>4</td>
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<td>0</td>
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<td>4</td>
<td>16</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>36</td>
</tr>
</tbody>
</table>

c. Graph both equations: y = f(x) and y = f(2x).

Scaffolding:
- Provide graphing calculators or utilities in order to allow all students to focus on the concept being developed in this example.
- For students that are ready for an extension, consider asking them to make a conjecture about an extreme case, such as h(x) = f(1000x) or a case that combines concepts from Lessons 17 and 18, such as p(x) = f(3x) - 4 or q(x) = f(3x + 6).
- Provide sentence frames to assist students in responding to part d. For example: “The graph of y = f(x) is similar to the graph of y = g(x) because _______. ...” and “The graph of y = f(x) is different from the graph of y = g(x) because _______. ...”

d. How does the graph of y = g(x) relate to the graph of y = f(x)?

The corresponding x-value of y = g(x) is half of the corresponding x-value of y = f(x) when g(x) = f(x); the points of the graph of g are \( \frac{1}{2} \) the distance to the y-axis as the corresponding points of the graph of f, which makes the graph of g appear to “shrink horizontally.”

e. How are the values of f related to the values of g?

For equal outputs of f and g, the input of g only has to be half as big as the input of f.
Discussion (5 minutes)

- A horizontal scaling of a graph with scale factor $\frac{1}{2}$ will “shrink” the original graph $y = f(x)$ horizontally by $\frac{1}{2}$ and correspond to the graph of the equation $y = f\left(\frac{1}{2}x\right)$ or $y = f(2x)$, i.e., the horizontal scaling of the graph of $y = f(x)$ with scale factor $k > 0$ is the graph of the equation $y = f\left(\frac{1}{k}x\right)$.

- In Example 1, what process could be used to find the value of $g(x)$ for any given number $x$, using only the graph of $y = f(x)$ (not the formula for $f(x)$)?
  - **Step 1:** Find $x$ on the $x$-axis.
  - **Step 2:** Multiply $x$ by 2 to find the number $2x$ on the $x$-axis.
  - **Step 3:** Find the value of $f$ at $2x$.
  - **Step 4:** Move parallel to the $x$-axis from the point found in Step 3 until directly over/under/on $x$. That point is $(x, g(x))$. [These steps are numbered and illustrated in the graph above for $x = 1$.]

Lightly erase the graph of $y = g(x)$ (already drawn from part (c)), and then go through the steps above to redraw it, picking out a few points to help students see that only the $x$-values are changing between corresponding points on the graph of $f$ and the graph of $g$. If you erased the graph lightly enough so that the “ghost” of the image is still there, students will see that you are redrawing the graph of $g$ over the original graph. Following the steps will give students a sense of how the points of the graph of $f$ are only “shrinking” in the $x$-values, not the $y$-values.

Many students might confuse a horizontal scaling with other types of transformations like dilations. In fact, a dilation with scale factor $\frac{1}{4}$ of the graph of $f$ in this example produces the exact same image as a horizontal scaling by $\frac{1}{2}$, but the correspondence between the points is different. Your goal in Grade 9 is to have students develop a “rigid” notion of what a vertical scaling means so that it can be profitably compared to dilation in Grades 10 and 11.

- Consider a function $f$, and a transformation of that function $h$, such that $h(x) = f\left(\frac{1}{k}x\right)$, how do the domain and range of $f$ relate to the domain and range of $h$?
  - **The range of both functions will be the same, but the domains may change.**

- What might the graph of $y = f(1,000x)$ look like?

- What might the graph of $y = f(1,000,000x)$ look like if it were graphed on the same Cartesian plane as the graphs of $f$ and $g$?

Let students go up to the board and draw their conjectures on the plane.

Discussion (5 minutes)

Students explore the horizontal scaling of the graph of $y = f(x)$ when the equation changes from $y = f(x)$ to $y = f\left(\frac{1}{k}x\right)$ for $k > 1$. In this case, students see that the graph of $f$ is horizontally “stretched” by a factor of $k$. In Example 2, the scale factor for $g$ is $k = 2$, or $g(x) = f\left(\frac{1}{2}x\right)$. 

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Example 2 (8 minutes)

Example 2

Let \( f(x) = x^2 \) and \( h(x) = f\left(\frac{1}{2}x\right) \), where \( x \) can be any real number.

a. Rewrite the formula for \( h \) in terms of \( x^2 \) (i.e., without using \( f(x) \) notation):

\[
h(x) = \left(\frac{1}{2}x\right)^2 \]

b. Complete the table of values for these functions.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = x^2 )</th>
<th>( h(x) = f\left(\frac{1}{2}x\right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>9</td>
<td>2.25</td>
</tr>
<tr>
<td>-2</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>0.25</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.25</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>2.25</td>
</tr>
</tbody>
</table>

c. Graph both equations: \( y = f(x) \) and \( y = f\left(\frac{1}{2}x\right) \).

Scaffolding:

- Provide graphing calculators or utilities in order to allow all students to focus on the concept being developed in this example.
- As an extension, ask students to make a conjecture about the appearance of \( m(x) = \left(\frac{1}{1000}x\right)^2 \).
d. How does the graph of \( y = f(x) \) relate to the graph of \( y = h(x) \)?

Since the corresponding \( x \)-value of \( y = h(x) \) is twice the corresponding \( x \)-value of \( y = f(x) \) when \( g(x) = f(x) \), the points of the graph of \( g \) are 2 times the distance to the \( y \)-axis as the corresponding points of the graph of \( f \), which makes the graph of \( g \) appear to “stretch horizontally.”

e. How are the values of \( f \) related to the values of \( h \)?

To get equal outputs of each function, the input of \( h \) has to be twice the input of \( f \).

A horizontal scale of a graph with scale factor 2 will "stretch" the original graph \( y = f(x) \) horizontally by 2 and correspond to the graph of the equation \( y = f \left( \frac{1}{2} x \right) \), i.e., the horizontal scale of the graph of \( y = f(x) \) with scale factor \( k > 0 \) is once again the graph of the equation \( y = f \left( \frac{1}{h} x \right) \). Follow the steps given in Discussion 1 to show students how to find the value \( h(x) \) on the Cartesian plane using only the graph of \( f \) (not the formula for \( f \)). Emphasize that only the \( y \)-values are being scaled. When comparing \( y = f(x) \) to \( y = f \left( \frac{1}{2} x \right) \), the range of both functions will be the same, but the domains may change. Ask students what the graph of \( f \) might look like after a horizontal scale with scale factor \( k = 10000 \). Let them draw their conjecture on the graph on the board. Then ask them what the equation of the resulting graph is.

Exercise 1 (6 minutes)

Before beginning these exercises, ask students to make a conjecture about the appearance of the graphs of \( f(x) = 2^x, g(x) = 2^{2x}, \) and \( h(x) = 2^{(-x)} \) either in writing or verbally with a partner. Use this as an opportunity to engage students further in MP.3 and informally assess understanding of the lesson thus far. Then have students discuss the following exercise in pairs. Discuss the answer as a class and ask students to reflect on their conjectures.

Exercise 1

Complete the table of values for the given functions.

a.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = 2^x )</th>
<th>( g(x) = 2^{2x} )</th>
<th>( h(x) = 2^{(-x)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-2)</td>
<td>(\frac{1}{4})</td>
<td>(\frac{1}{16})</td>
<td>(4)</td>
</tr>
<tr>
<td>(-1)</td>
<td>(\frac{1}{2})</td>
<td>(\frac{1}{4})</td>
<td>(2)</td>
</tr>
<tr>
<td>(0)</td>
<td>(1)</td>
<td>(1)</td>
<td>(1)</td>
</tr>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>(4)</td>
<td>(\frac{1}{2})</td>
</tr>
<tr>
<td>(2)</td>
<td>(4)</td>
<td>(16)</td>
<td>(\frac{1}{4})</td>
</tr>
</tbody>
</table>

Scaffolding:

- Provide graphing calculators or utilities in order to allow all students to focus on the concept being developed in this example.
b. Label each of the graphs with the appropriate functions from the table.

![Graph](image)


c. Describe the transformation that takes the graph of \( y = f(x) \) to the graph of \( y = g(x) \).

The graph of \( y = g(x) \) is a horizontal scale with scale factor \( \frac{1}{2} \) of the graph of \( y = f(x) \).

d. Consider \( y = f(x) \) and \( y = h(x) \). What does negating the input do to the graph of \( f \)?

The graph of \( h \) is a reflection over the \( y \)-axis of the graph of \( f \).

e. Write the formula of an exponential function whose graph would be a horizontal stretch relative to the graph of \( g \).

Answers will vary. Example: \( y = 2^{(0.5x)} \).

Example 3 (6 minutes)

Example 3

a. Look at the graph of \( y = f(x) \) for the function \( f(x) = x^2 \) in Example 1 again. Would we see a difference in the graph of \( y = g(x) \) if \(-2\) was used as the scale factor instead of \(2\)? If so, describe the difference. If not, explain why not.

There would be no difference. The function involves squaring the value within the parentheses, so the graph of \( y = f(2x) \) and the graph of \( y = f(-2x) \) both will be the same set as the graph of \( y = g(x) \), but both correspond...
to different transformations: The first is a horizontal scaling with scale factor \( \frac{1}{2} \) and the second is a horizontal scaling with scale factor \( \frac{1}{2} \) and a reflection across the y-axis.

b. A reflection across the y-axis takes the graph of \( y = f(x) \) for the function \( f(x) = x^2 \) back to itself. Such a transformation is called a **reflection symmetry**. What is the equation for the graph of the reflection symmetry of the graph of \( y = f(x) \)?

\[ y = f(-x). \]

Tell students that if a function satisfies the equation \( f(x) = f(-x) \) for every number \( x \) in the domain of \( f \), it is called an **even function**. A consequence of an even function is that its graph is symmetrical with respect to the y-axis. Furthermore, the graph of \( f(x) = x^2 \) is symmetrical across the y-axis. A reflection across the y-axis does not change the graph.

c. Deriving the answer to the following question is fairly sophisticated; do only if you have time: In Lessons 17 and 18, we used the function \( f(x) = |x| \) to examine the graphical effects of transformations of a function. Here in Lesson 19, we use the function \( f(x) = x^2 \) to examine the graphical effects of transformations of a function. Based on the observations you made while graphing, why would using \( f(x) = x^2 \) be a better option than using the function \( f(x) = |x| \)?

*Not all of the effects of multiplying the input of a function are as visible with an absolute function as it is with a quadratic function. For example, the graph of \( y = 2|x| \) is the same as \( y = |2x| \). Therefore, it is easier to see the effect of multiplying a value to the input of a function by using a quadratic function than it is by using the absolute value function.*

**Closing (2 minutes)**

Discuss how the horizontal scaling by a scale factor of \( k \) of the graph of a function \( y = f(x) \) corresponds to changing the equation of the graph from \( y = f(x) \) to \( y = f \left( \frac{x}{k} \right) \). Ask students to describe the effect of the four cases of \( k \) shown below, either in writing or with a partner. Use this as an opportunity to informally assess understanding of the ideas in this lesson.

1. \( k > 1 \)
2. \( 0 < k < 1 \)
3. \( -1 < k < 0 \)
4. \( k < -1 \)

**Exit Ticket (5 minutes)**

Ensure that students complete the Exit Ticket independently, in order to provide evidence of each student’s understanding of the lesson.
Lesson 19: Four Interesting Transformations of Functions

Exit Ticket

Let $f(x) = x^2$, $g(x) = (3x)^2$, and $h(x) = \left(\frac{1}{3} x\right)^2$, where $x$ can be any real number. The graphs above are of $y = f(x)$, $y = g(x)$, and $y = h(x)$.

1. Label each graph with the appropriate equation.

2. Describe the transformation that takes the graph of $y = f(x)$ to the graph of $y = g(x)$. Use coordinates of each to illustrate an example of the correspondence.
3. Describe the transformation that takes the graph of \( y = f(x) \) to the graph of \( y = h(x) \). Use coordinates to illustrate an example of the correspondence.
Exit Ticket Sample Solutions

Let \( f(x) = x^2 \), \( g(x) = (3x)^2 \), and \( h(x) = \left(\frac{1}{3}x\right)^2 \), where \( x \) can be any real number. The graphs above are of \( y = f(x) \), \( y = g(x) \), and \( y = h(x) \).

1. Label each graph with the appropriate equation.
   See graph.

2. Describe the transformation that takes the graph of \( y = f(x) \) to the graph of \( y = g(x) \). Use coordinates of each to illustrate an example of the correspondence.

   The graph of \( y = g(x) \) is a horizontal shrink of the graph of \( y = f(x) \) with scale factor \( \frac{1}{3} \). The corresponding \( x \)-value of \( y = g(x) \) is one-third the corresponding \( x \)-value of \( y = f(x) \) when \( g(x) = f(x) \). This can be illustrated with the coordinate \((1, 9)\) on \( g(x) \) and the coordinate \((3, 9)\) on \( f(x) \).

3. Describe the transformation that takes the graph of \( y = f(x) \) to the graph of \( y = h(x) \). Use coordinates to illustrate an example of the correspondence.

   The graph of \( h(x) \) is a horizontal stretch of the graph of \( f(x) \) with scale factor \( 3 \). The corresponding \( x \)-value of \( y = h(x) \) is three times the corresponding \( x \)-value of \( y = f(x) \) when \( h(x) = f(x) \). This can be illustrated with the coordinate \((1, 1)\) on \( f(x) \) and the coordinate \((3, 1)\) on \( h(x) \).
Problem Set Sample Solutions

Let \( f(x) = x^2 \), \( g(x) = 2x^2 \), and \( h(x) = (2x)^2 \), where \( x \) can be any real number. The graphs above are of the functions \( y = f(x) \), \( y = g(x) \), and \( y = h(x) \).

1. Label each graph with the appropriate equation.
   
   See graph.

2. Describe the transformation that takes the graph of \( y = f(x) \) to the graph of \( y = g(x) \). Use coordinates to illustrate an example of the correspondence.

   The graph of \( y = g(x) \) is a vertical stretch of the graph of \( y = f(x) \) by scale factor 2; for a given \( x \)-value, the value of \( g(x) \) is twice as much as the value of \( f(x) \).

   OR

   The graph of \( y = g(x) \) is a horizontal shrink of the graph of \( y = f(x) \) by scale factor \( \frac{1}{\sqrt{2}} \); it takes \( \frac{1}{\sqrt{2}} \) times the input for \( y = g(x) \) as compared to \( y = f(x) \) to yield the same output.

3. Describe the transformation that takes the graph of \( y = f(x) \) to the graph of \( y = h(x) \). Use coordinates to illustrate an example of the correspondence.

   The graph of \( y = h(x) \) is a horizontal shrink of the graph of \( y = f(x) \) by scale factor \( \frac{1}{2} \); it takes \( \frac{1}{2} \) the input for \( y = h(x) \) as compared to \( y = f(x) \) to yield the same output.

   OR

   The graph of \( y = h(x) \) is a vertical stretch of the graph of \( y = f(x) \) by scale factor 4; for a given \( x \)-value, the value of \( h(x) \) is four times as much as the value of \( f(x) \).
Lesson 20: Four Interesting Transformations of Functions

Student Outcomes

- Students apply their understanding of transformations of functions and their graphs to piecewise functions.

Lesson Notes

In Lessons 17–19 students study translations and scalings of functions and their graphs. In Lesson 20, emphasis continues on F-BF.B.3, but F-IF.C.7b comes back into focus as these transformations are applied to piecewise-defined functions. Students should become comfortable visualizing how the graph of a transformed piecewise function will relate to the graph of the original piecewise function.

Classwork

Opening Exercise (8 minutes)

Have students work individually or in pairs to complete the Opening Exercise. This exercise highlights MP.7 as it calls on students to interpret the meaning of $k$ in the context of a graph.

Opening Exercise

Fill in the blanks of the table with the appropriate heading or descriptive information.

<table>
<thead>
<tr>
<th>Graph of $y = f(x)$</th>
<th>Vertical</th>
<th>Horizontal</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Translate</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$y = f(x) + k$</td>
<td>$k &gt; 0$</td>
<td>$k &gt; 0$</td>
</tr>
<tr>
<td></td>
<td>Translate up by $</td>
<td>k</td>
</tr>
<tr>
<td></td>
<td>$k &lt; 0$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Translate down by $</td>
<td>k</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$k &lt; 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Translate left by $</td>
</tr>
<tr>
<td><strong>Scale by scale factor $k$</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$y = kf(x)$</td>
<td>$k &gt; 1$</td>
<td>$k &gt; 1$</td>
</tr>
<tr>
<td></td>
<td>Vertical stretch by a factor of $</td>
<td>k</td>
</tr>
<tr>
<td></td>
<td>$0 &lt; k &lt; 1$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Vertical shrink by a factor of $</td>
<td>k</td>
</tr>
<tr>
<td></td>
<td>$-1 &lt; k &lt; 0$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Vertical shrink by a factor of $</td>
<td>k</td>
</tr>
<tr>
<td></td>
<td>$k &lt; -1$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Vertical stretch by a factor of $</td>
<td>k</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$k &lt; -1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Horizontal stretch by a factor of $</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$-1 &lt; k &lt; 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Horizontal shrink by a factor of $</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$k &lt; -1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Horizontal stretch by a factor of $</td>
</tr>
</tbody>
</table>
In Lesson 15, we discovered how the absolute value function can be written as a piecewise function. Example 1 and the associated exercises are intended to help students reexamine how piecewise functions behave; additionally, this example revisits the cell phone example from Lesson 16.

Example 1 (8 minutes)

Example 1
Let’s revisit the cell phone example from Lesson 16. Recall that one of the plans costs $45 with 80 free texts, then $0.25 for each text after the first 80. The piecewise function, \( g(x) \), where \( g(x) \) represents the cost of the plan for \( x \) texts is shown below. Describe in words how this function models the plan and how to graph this piecewise function.

\[
g(x) = \begin{cases} 
45, & 0 \leq x \leq 80 \\
0.25x + 25, & x > 80 
\end{cases}
\]

The function returns a constant value of 45 for the first 80 texts since they are included in the plan. For more than 80 texts, the function adds 0.25 per text, with the y-intercept chosen so that when \( x \) is 80, the function returns a value of 45. To graph this function, I would graph the line \( y = 45 \) for \( x \)-values from 0 to 80, and then I would graph the line \( y = 0.25x + 25 \) for \( x \)-values greater than 80.

Ask students to consider questions such as, “What would the graph of \( 2g(x) \) look like? What would the graph of \( g(x) + 10 \) look like?” Encourage students to make and justify conjectures based on what they learned in Lessons 17 – 19, either in writing or with a partner, then share as a class.

Exercises 1–2 (8 minutes)
Ask students to try graphing these exercises in small groups. Circulate to informally assess their understanding of the process.

Exercises 1–2
1. Describe how to graph the following piecewise function. Then graph \( y = f(x) \) below.

\[
f(x) = \begin{cases} 
-3x - 3, & x \leq -2 \\
0.5x + 4, & -2 < x < 2 \\
-2x + 9, & x \geq 2 
\end{cases}
\]

The function \( f \) can be graphed of as the line \( y = -3x - 3 \) for \( x \)-values less than or equal to \(-2\), the graph of the line \( y = 0.5x + 4 \) for \( x \)-values greater than \(-2\) and less than \(2\), and the graph of the line \( y = -2x + 9 \) for \( x \)-values greater than \(2\) or equal to \(2\).
2. Using the graph of \( f \) below, write a formula for \( f \) as a piecewise function.

\[
f(x) = \begin{cases} 
0.5x - 0.5, & -7 \leq x \leq -3 \\
-2, & -3 < x < 1 \\
x - 3, & 1 \leq x \leq 4 \\
5 - x, & 4 < x \leq 7 
\end{cases}
\]

or

\[
f(x) = \begin{cases} 
0.5x - 0.5, & -7 \leq x \leq -3 \\
-2, & -3 < x < 1 \\
-|x - 4| + 1, & 1 \leq x \leq 7 
\end{cases}
\]
Example 2 (10 minutes)

Students translate and scale the graph of a piecewise function; allow students to work with a partner. Following parts b and c, ask students to summarize how they performed each translation to a partner. Listen in to the conversations to assess understanding of the application of transformations to piecewise-defined functions.

Example 2

The graph \( y = f(x) \) of a piecewise function \( f \) is shown. The domain of \( f \) is \(-5 \leq x \leq 5\), and the range is \(-1 \leq y \leq 3\).

a. Mark and identify four strategic points helpful in sketching the graph of \( y = f(x) \).

\((-5, -1), (-1, 1), (3, 1), \) and \((5, 3)\)

b. Sketch the graph of \( y = 2f(x) \) and state the domain and range of the transformed function. How can you use part (a) to help sketch the graph of \( y = 2f(x) \)?

**Domain:** \(-5 \leq x \leq 5\), **Range:** \(-2 \leq y \leq 6\). For every point \((x, y)\) in the graph of \( f(x) \), there is a point \((x, 2y)\) on the graph of \( y = 2f(x) \). The four strategic points can be used to determine the line segments in the graph of \( y = 2f(x) \) by graphing points with the same original \( x \)-coordinate and 2 times the original \( y \)-coordinate \((-5, -2), (-1, 2), (3, 2), \) and \((5, 6)\).
c. A horizontal scaling with scale factor $\frac{1}{2}$ of the graph of $y = f(x)$ is the graph of $y = f(2x)$. Sketch the graph of $y = f(2x)$ and state the domain and range. How can you use the points identified in part (a) to help sketch $y = f(2x)$?

**Domain:** $-2.5 \leq x \leq 2.5$, **range:** $-1 \leq y \leq 3$. For every point $(x, y)$ in the graph of $f(x)$, there is a point $\left(\frac{x}{2}, y\right)$ on the graph of $y = f(2x)$. The four strategic points can be used to determine the line segments in the graph of $y = f(2x)$ by graphing points with one-half the original $x$-coordinate and the original $y$-coordinate $\left((-2.5, -1), (-0.5, 1), (1.5, 1), (2.5, 3)\right)$.

Exercises 3–4 (5 minutes)

Ask students to work on these exercises independently. Circulate to informally assess understanding.

### Exercises 3–4

3. How does the range of $f$ in Example 2 compare to the range of a transformed function $g$, where $g(x) = kf(x)$, when $k > 1$?

For every point $(x, y)$ in the graph of $y = f(x)$, there is a point $(x, ky)$ in the graph of $y = kf(x)$, where the number $ky$ is a multiple of each $y$. For values of $k > 1$, $y = kf(x)$ is a vertical scaling that appears to stretch the graph of $y = f(x)$. The original range, $-1 \leq y \leq 3$ for $y = f(x)$ becomes $-1k \leq y \leq 3k$ for the function $y = kf(x)$.

4. How does the domain of $f$ in Example 2 compare to the domain of a transformed function $g$, where $g(x) = f\left(\frac{1}{k}x\right)$, when $0 < k < 1$? (Hint: How does a graph shrink when it is horizontally scaled by a factor $k$?)

For every point $(x, y)$ in the graph of $y = f(x)$, there is a point $\left(kx, y\right)$ in the graph of $y = f\left(\frac{1}{k}x\right)$. For values of $0 < k < 1$, $y = f\left(\frac{1}{k}x\right)$ is a horizontal scaling by a factor $k$ that appears to shrink the graph of $y = f(x)$. This means the original domain, $-5 \leq x \leq 5$ for $y = f(x)$ becomes $-5k \leq x \leq 5k$ for the function $y = f\left(\frac{1}{k}x\right)$.
Closing (2 minutes)

Ask students to write or explain to a neighbor the key ideas of the lesson. These are some key ideas to look for:

- The transformations that translate and scale familiar functions, like the absolute value function, also apply to piecewise functions and to any function in general.
- By focusing on strategic points in the graph of a piecewise function, we can translate and scale the entire graph of the function by manipulating the coordinates of those few points.

Exit Ticket (4 minutes)

Ensure that students complete the Exit Ticket independently, in order to provide evidence of each student’s understanding of the lesson.
Exit Ticket

The graph of a piecewise function \( f \) is shown below.

Let \( p(x) = f(x - 2) \), \( q(x) = \frac{1}{2} f(x - 2) \), and \( r(x) = \frac{1}{2} f(x - 2) + 3 \).

Graph \( y = p(x) \), \( y = q(x) \), and \( y = r(x) \) on the same set of axes as the graph of \( y = f(x) \).
Exit Ticket Sample Solutions

The graph of a piecewise function $f$ is shown below.

Let $p(x) = f(x - 2)$, $q(x) = \frac{1}{2} f(x - 2)$, and $r(x) = \frac{1}{2} f(x - 2) + 3$.

Graph $y = p(x)$, $y = q(x)$, and $y = r(x)$ on the same set of axes as the graph of $y = f(x)$. 

[Graph Image]
Problem Set Sample Solutions

1. Suppose the graph of \( f \) is given. Write an equation for each of the following graphs after the graph of \( f \) has been transformed as described. Note that the transformations are not cumulative.
   
   a. Translate 5 units upward.
      \[
      y = f(x) + 5
      \]
   
   b. Translate 3 units downward.
      \[
      y = f(x) - 3
      \]
   
   c. Translate 2 units right.
      \[
      y = f(x - 2)
      \]
   
   d. Translate 4 units left.
      \[
      y = f(x + 4)
      \]
   
   e. Reflect about the \( x \)-axis.
      \[
      y = -f(x)
      \]
   
   f. Reflect about the \( y \)-axis.
      \[
      y = f(-x)
      \]
   
   g. Stretch vertically by a factor of 2.
      \[
      y = 2f(x)
      \]
   
   h. Shrink vertically by a factor of \( \frac{1}{3} \).
      \[
      y = \frac{1}{3}f(x)
      \]

   i. Shrink horizontally by a factor of \( \frac{1}{3} \).
      \[
      y = f(3x)
      \]
   
   j. Stretch horizontally by a factor of 2.
      \[
      y = f\left(\frac{1}{2}x\right)
      \]

2. Explain how the graphs of the equations below are related to the graph of \( y = f(x) \).
   
   a. \( y = 5f(x) \)
      The graph is a vertical stretch of \( y = f(x) \) by a factor of 5.
b. \[ y = f(x - 4) \]

*The graph of \( y = f(x) \) is translated right 4 units.*

c. \[ y = -2f(x) \]

*The graph is a vertical stretch of \( y = f(x) \) by a factor of 2 and reflected about the \( x \)-axis.*

d. \[ y = f(3x) \]

*The graph is a horizontal shrink of \( y = f(x) \) by a factor of \( \frac{1}{3} \).*

e. \[ y = 2f(x) - 5 \]

*The graph is a vertical stretch of \( y = f(x) \) by a factor of 2 and translated down 5 units.*

3. The graph of the equation \( y = f(x) \) is provided below. For each of the following transformations of the graph, write a formula (in terms of \( f \)) for the function that is represented by the transformation of the graph of \( y = f(x) \). Then draw the transformed graph of the function on the same set of axes as the graph of \( y = f(x) \).

- A translation 3 units left and 2 units up.

\[ p(x) = f(x + 3) + 2 \]
b. A vertical stretch by a scale factor of 3.

\[ q(x) = 3f(x) \]

c. A horizontal shrink by a scale factor of \( \frac{1}{2} \).

\[ r(x) = f(2x) \]

4. Reexamine your work on Example 2 and Exercises 3 and 4 from this lesson. Parts (b) and (c) of Example 2 asked how the equations \( y = 2f(x) \) and \( y = f(2x) \) could be graphed with the help of the strategic points found in (a). In this problem, we investigate whether it is possible to determine the graphs of \( y = 2f(x) \) and \( y = f(2x) \) by working with the piecewise-linear function \( f \) directly.

a. Write the function \( f \) in Example 2 as a piecewise-linear function.

\[
f(x) = \begin{cases} 
 0.5x + 1.5, & -5 \leq x \leq -1 \\
 1, & -1 < x < 3 \\
 x - 2, & 3 \leq x \leq 5 
\end{cases}
\]
Lesson 20: Four Interesting Transformations of Functions

b. Let \( g(x) = 2f(x) \). Use the graph you sketched in Example 2, part (b) of \( y = 2f(x) \) to write the formula for the function \( g \) as a piecewise-linear function.

\[
g(x) = \begin{cases} 
  x + 3, & -5 \leq x \leq -1 \\
  2, & -1 < x < 3 \\
  2x - 4, & 3 \leq x \leq 5 
\end{cases}
\]

c. Let \( h(x) = f(2x) \). Use the graph you sketched in Example 2, part (c) of \( y = f(2x) \) to write the formula for the function \( h \) as a piecewise-linear function.

\[
h(x) = \begin{cases} 
  x + 1.5, & -2.5 \leq x \leq -0.5 \\
  1, & -0.5 < x < 1.5 \\
  2x - 2, & 1.5 \leq x \leq 2.5 
\end{cases}
\]

d. Compare the piecewise linear functions \( g \) and \( h \) to the piecewise linear function \( f \). Did the expressions defining each piece change? If so, how? Did the domains of each piece change? If so how?

**Function g:** Each piece of the formula for \( g \) is 2 times the corresponding piece of the formula for \( f \). The domains are the same.

**Function h:** Each piece of the formula for \( h \) is found by substituting 2x in for \( x \) in the corresponding piece of the formula for \( f \). The length of each interval in the domain of \( h \) is \( \frac{1}{2} \) the length of the corresponding interval in the domain of \( f \).
1. Given \( h(x) = |x + 2| - 3 \) and \( g(x) = -|x| + 4 \).

   a. Describe how to obtain the graph of \( g \) from the graph of \( a(x) = |x| \) using transformations.

   b. Describe how to obtain the graph of \( h \) from the graph of \( a(x) = |x| \) using transformations.

   c. Sketch the graphs of \( h(x) \) and \( g(x) \) on the same coordinate plane.

   d. Use your graphs to estimate the solutions to the equation:
      \[ |x + 2| - 3 = -|x| + 4 \]
      Explain how you got your answer.

   e. Were your estimations you made in part (d) correct? If yes, explain how you know. If not explain why not.
2. Let $f$ and $g$ be the functions given by $f(x) = x^2$ and $g(x) = x|x|$.
   
   a. Find $f\left(\frac{1}{3}\right)$, $g(4)$, and $g(-\sqrt{3})$.
   
   b. What is the domain of $f$?
   
   c. What is the range of $g$?
   
   d. Evaluate $f(-67) + g(-67)$.
   
   e. Compare and contrast $f$ and $g$. How are they alike? How are they different?
   
   f. Is there a value of $x$, such that $f(x) + g(x) = -100$? If so, find $x$. If not, explain why no such value exists.
   
   g. Is there a value of $x$ such that $f(x) + g(x) = 50$? If so, find $x$. If not, explain why no such value exists.
3. A boy bought 6 guppies at the beginning of the month. One month later the number of guppies in his tank had doubled. His guppy population continued to grow in this same manner. His sister bought some tetras at the same time. The table below shows the number of tetras, $t$, after $n$ months have passed since they bought the fish.

<table>
<thead>
<tr>
<th>$n$, months</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$, tetras</td>
<td>8</td>
<td>16</td>
<td>24</td>
<td>32</td>
</tr>
</tbody>
</table>

a. Create a function $g$ to model the growth of the boy’s guppy population, where $g(n)$ is the number of guppies at the beginning of each month, and $n$ is the number of months that have passed since he bought the 6 guppies. What is a reasonable domain for $g$ in this situation?

b. How many guppies will there be one year after he bought the 6 guppies?

c. Create an equation that could be solved to determine how many months after he bought the guppies there will be 100 guppies.

d. Use graphs or tables to approximate a solution to the equation from part (c). Explain how you arrived at your estimate.
e. Create a function, \( t \), to model the growth of the sister’s tetra population, where \( t(n) \) is the number of tetras after \( n \) months have passed since she bought the tetras.

f. Compare the growth of the sister’s tetra population to the growth of the guppy population. Include a comparison of the average rate of change for the functions that model each population’s growth over time.

g. Use graphs to estimate the number of months that will have passed when the population of guppies and tetras will be the same.
h. Use graphs or tables to explain why the guppy population will eventually exceed the tetra population even though there were more tetras to start with.

i. Write the function \( g(n) \) in such a way that the percent increase in the number of guppies per month can be identified. Circle or underline the expression representing percent increase in number of guppies per month.
4. Regard the solid dark equilateral triangle as Figure 0. Then, the first figure in this sequence is the one composed of three dark triangles, the second figure is the one composed of nine dark triangles, and so on.

![Triangle Figures](image)

**Figure 0**  **Figure 1**  **Figure 2**  **Figure 3**  **Figure 4**

a. How many dark triangles are in each figure? Make a table to show this data.

| $n$ (Figure Number) | | | |
|---------------------|----------------|
| $T$ (# of dark triangles) | | | |

b. Describe in words how, given the number of dark triangles in a figure, to determine the number of dark triangles in the next figure.

c. Create a function that models this sequence. What is the domain of this function?

d. Suppose the area of the solid dark triangle in Figure 0 is 1 square meter. The areas of one dark triangle in each figure form a sequence. Create an explicit formula that gives the area of just one of the dark triangles in the $n^{th}$ figure in the sequence.
e. The sum of the areas of all the dark triangles in Figure 0 is \(1\) \(m^2\); there is only one triangle in this case. The sum of the areas of all the dark triangles in Figure 1 is \(\frac{3}{4}\) \(m^2\). What is the sum of the areas of all the dark triangles in the \(n^{th}\) figure in the sequence? Is this total area increasing or decreasing as \(n\) increases?

f. Let \(P(n)\) be the sum of the perimeters of the all dark triangles in the \(n^{th}\) figure in the sequence of figures. There is a real number \(k\) so that:

\[
P(n + 1) = kP(n)
\]

is true for each positive whole number \(n\). What is the value of \(k\)?
5. The graph of a piecewise function \( f \) is shown to the right. The domain of \( f \) is \(-3 \leq x \leq 3\).

a. Create an algebraic representation for \( f \). Assume that the graph of \( f \) is composed of straight line segments.

b. Sketch the graph of \( y = 2f(x) \) and state the domain and range.
c. Sketch the graph of \( y = f(2x) \) and state the domain and range.

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
x & -6 & -5 & -4 & -3 & -2 & -1 & 0 \\
\hline
y & -6 & -5 & -4 & -3 & -2 & -1 & 0 \\
\hline
\end{array}
\]

\( \text{Domain: } (-\infty, \infty) \)  
\( \text{Range: } (0, \infty) \)

d. How does the range of \( y = f(x) \) compare to the range of \( y = kf(x) \), where \( k > 1 \)?

e. How does the domain of \( y = f(x) \) compare to the domain of \( y = f(kx) \), where \( k > 1 \)?
### A Progression Toward Mastery

<table>
<thead>
<tr>
<th>Assessment Task Item</th>
<th>STEP 1</th>
<th>STEP 2</th>
<th>STEP 3</th>
<th>STEP 4</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1</strong> a–b F-BF.B.3</td>
<td>Student answer is missing or entirely incorrect.</td>
<td>Student describes the transformations partially correctly.</td>
<td>Student describes the transformations correctly, but there may be some minor misuse or omission of appropriate vocabulary.</td>
<td>Student describes the transformations clearly and correctly AND uses appropriate vocabulary.</td>
</tr>
<tr>
<td>c–e A-REI.D.11 F-BF.B.3</td>
<td>Student creates sketches that do not resemble absolute value functions. OR Student is unable to use the graphs to estimate the solutions to the equation. Student may or may not have arrived at correct solutions of the equation via another method such as trial and error.</td>
<td>Student creates sketches that resemble the graph of an absolute value function but are inaccurate. Student shows evidence of using the intersection point of the graphs to find the solution but is unable to confirm his or her solution points; therefore, the conclusion in part (e) is inconsistent with the intersection points.</td>
<td>Student creates sketches that are accurate with no more than one minor error; the student shows evidence of using the intersection points to find the solutions to the equation. The conclusion in part (e) is consistent with the estimated solutions but may have one error. Student communication is clear but could include more appropriate use of vocabulary or more detail.</td>
<td>Student creates sketches that are accurate and solutions in part (d) match the x-coordinates of the intersection points. The student’s explanation for part (d) reflects an understanding that the process is analogous to solving the system ( y = h(x) ) and ( y = g(x) ). The work shown in part (e) supports his or her conclusion that estimates were or were not solutions and includes supporting explanation using appropriate vocabulary.</td>
</tr>
<tr>
<td><strong>2</strong> a F-IF.A.2</td>
<td>Student provides no correct answers.</td>
<td>Student provides only one correct answer.</td>
<td>Student provides two correct answers.</td>
<td>Student provides correct answers for all three items.</td>
</tr>
<tr>
<td></td>
<td>b – c</td>
<td>Neither domain nor range is correct.</td>
<td>One of the two is identified correctly, or the student has reversed the ideas, giving the range of ( f ) when asked for domain of ( f ), and the domain of ( g ) when asked for the range of ( g ).</td>
<td>Both domain and range are correct but notation may contain minor errors.</td>
</tr>
<tr>
<td>---</td>
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</tr>
<tr>
<td>F-IF.A.1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| d | Student makes a major error or omission in evaluating the expression (e.g., doesn’t substitute -67 into \( f \) or \( g \)) | Student makes one or more errors in evaluating the expression. | Student evaluates the expression correctly but work to support the answer is limited, or there is one minor error present. | Student evaluates the expression correctly and shows the work to support the answer. |
| F-IF.A.2 |   |   |   |   |   |

| e | Student makes little or no attempt to compare the two functions. | Student comparison does not note the similarity of the two functions yielding identical outputs for positive inputs and opposite outputs for negative inputs; it may be limited to superficial features, such as one involves squaring and the other contains an absolute value. | Student recognizes that the two functions are equal for \( x = 0 \) and positive \( x \)-values but may not clearly articulate that the two functions are opposites when \( x \) is negative. | Student clearly describes that the two functions yield identical outputs for positive inputs and for an input of \( x = 0 \), and opposite outputs for negative inputs. |
| F-IF.A.1 F-IF.A.2 F-IF.C.7a |   |   |   |   |   |

| f | Student provides an incorrect conclusion. OR Student makes little or no attempt to answer. | Student identifies that there is no solution but provides little or no supporting work or explanation. | Student identifies that there is no solution and provides an explanation, but the explanation is limited or contains minor inconsistencies or errors. | Student identifies that there is no solution and provides an explanation and/or work that clearly supports valid reasoning. |
| F-IF.A.1 F-IF.A.2 |   |   |   |   |   |

| g | Student provides an incorrect conclusion. OR Student makes little or no attempt to answer. | Student identifies that \( x = 5 \) is a solution but provides little or no supporting work or explanation. | Student identifies that \( x = 5 \) is a solution and provides an explanation, but the explanation is limited or contains minor inconsistencies/ errors. | Student identifies that \( x = 5 \) is a solution and provides an explanation and/or work that clearly supports valid reasoning. |
| F-IF.A.1 F-IF.A.2 |   |   |   |   |   |

<p>| 3 | a | Student does not provide an exponential function. OR Student provides an exponential function that does not model the data, and the domain is incorrect or omitted. | Student provides a correct exponential function, but the domain is incorrect or omitted. OR Student provides an exponential function that does not model the data but correctly identifies the domain in this situation. | Student has made only minor errors in providing an exponential function that models the data and a domain that fits the situation. | Student provides a correct exponential function and identifies the domain to fit the situation. |
| A-CED.A.1 F-BF.A.1a F-IF.B.5 |   |   |   |   |   |</p>
<table>
<thead>
<tr>
<th></th>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>b</strong></td>
<td><strong>F-IF.A.2</strong></td>
<td>Student gives an incorrect answer with no supporting calculations.</td>
<td>Student gives an incorrect answer, but the answer is supported with the student’s function from part (a).</td>
<td>Student has a minor calculation error in arriving at the answer. Student provides supporting work.</td>
</tr>
<tr>
<td><strong>c</strong></td>
<td><strong>F-BF.A.1a</strong></td>
<td>Student provides no equation or gives an equation that does not demonstrate understanding of what is required to solve the problem described.</td>
<td>Student sets up an incorrect equation that demonstrates limited understanding of what is required to solve the problem described.</td>
<td>Student provides a correct answer but then simplifies it into an incorrect equation. OR Student has a minor error in the equation given but demonstrates substantial understanding of what is required to solve the problem.</td>
</tr>
<tr>
<td><strong>d</strong></td>
<td><strong>F-IF.A.2</strong></td>
<td>Student provides an equation or graph that does not reflect the correct data. OR Student fails to provide an equation or graph.</td>
<td>Student provides a correct graph or table, but the answer to the question is either not given or incorrect.</td>
<td>Student provides a correct table or graph, but the answer is 4 months with an explanation that the 100 mark occurs during the 4th month.</td>
</tr>
<tr>
<td><strong>e</strong></td>
<td><strong>F-BF.A.1a</strong></td>
<td>Student does not provide a linear function.</td>
<td>Student provides a function that is linear but does not reflect data.</td>
<td>Student provides a correct linear function, but the function is either simplified incorrectly or does not use the notation, ( f(n) ).</td>
</tr>
<tr>
<td><strong>f</strong></td>
<td><strong>A-CED.A.2</strong></td>
<td>Student makes a partially correct but incomplete comparison of growth rates that does not include or incorrectly applies the concept of average rate of change.</td>
<td>Student makes a correct comparison of growth rates that includes an analysis of the rate of change of each function. However, student’s communication contains minor errors or misuse of mathematical terms.</td>
<td>Student identifies that the Guppies’ population will increase at a faster rate and provides a valid explanation that includes an analysis of the rate of change of each function.</td>
</tr>
<tr>
<td><strong>g</strong></td>
<td><strong>A-REI.D.11</strong></td>
<td>Student provides correct graphs but is unable to arrive at a correct answer from the graphs. OR Student’s graphs are incomplete or incorrect, but the student arrives at an answer based on sound reasoning.</td>
<td>Student provides graphs that contain minor imprecisions and therefore arrives at an answer that is supportable by the graphs but incorrect.</td>
<td>Student provides correct graphs and arrives at an answer that is supportable by the graphs and correct.</td>
</tr>
<tr>
<td>4</td>
<td>a–c</td>
<td>F-BF.A.1a</td>
<td>F-IF.A.3</td>
<td>F-LE.A.1</td>
</tr>
<tr>
<td>---</td>
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<tr>
<td></td>
<td>i</td>
<td>A-SSE.B.3c</td>
<td>Student does not provide an exponential function that shows percent increase.</td>
<td>Student writes an exponential function that uses an incorrect version of the growth factor, such as 0.02, 2%, 20%, or 0.20.</td>
</tr>
<tr>
<td></td>
<td>d</td>
<td>F-BF.A.1a</td>
<td>F-LE.A.1</td>
<td>F-LE.A.2</td>
</tr>
<tr>
<td></td>
<td>e</td>
<td>F-BF.A.1a</td>
<td>F-LE.A.1</td>
<td>F-LE.A.2</td>
</tr>
<tr>
<td></td>
<td>f</td>
<td>F-BF.A.1a</td>
<td>Student provides little or no evidence of understanding how to</td>
<td>Student identifies the incorrect value of ( k ) or is not provided, but</td>
</tr>
<tr>
<td>F-LE.A.1</td>
<td>determine the perimeter of the dark triangles nor how to recognize the common factor between two successive figures’ perimeter.</td>
<td>solution shows some understanding of how to determine the perimeter of the dark triangles.</td>
<td>evidence of student thinking (correct table, graph, marking on diagram, or calculations) that shows how he or she arrived at the solution.</td>
<td></td>
</tr>
<tr>
<td>---</td>
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<td>---</td>
<td>---</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>a</td>
<td>F-BF.A.1a</td>
<td>Student does not provide a piecewise definition of the function and/or more than two expressions in the answer are incorrect.</td>
<td></td>
</tr>
<tr>
<td>b–c</td>
<td>F-BF.B.3</td>
<td>Student provides graphs that contain major errors; domain and range are missing or are inconsistent with the graphs.</td>
<td>Student provides a graph for (b) that would be correct for (c) and vice versa.</td>
<td></td>
</tr>
<tr>
<td>d–e</td>
<td>F-BF.B.3</td>
<td>Both explanations and solutions are incorrect or have major conceptual errors (e.g., confusing domain and range).</td>
<td>Student answers contain more than one minor error.</td>
<td></td>
</tr>
</tbody>
</table>

**Module 3:** Linear and Exponential Functions

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1. Given \( h(x) = |x + 2| - 3 \) and \( g(x) = -|x| + 4 \).

   a. Describe how to obtain the graph of \( g \) from the graph of \( a(x) = |x| \) using transformations.

   \[ \text{To obtain the graph of } g, \text{ reflect the graph of } a \text{ about the x-axis and translate this graph up 4 units.} \]

   b. Describe how to obtain the graph of \( h \) from the graph of \( a(x) = |x| \) using transformations.

   \[ \text{To obtain the graph of } h, \text{ translate the graph of } a 2 \text{ units to the left and 3 units down.} \]

   c. Sketch the graphs of \( h(x) \) and \( g(x) \) on the same coordinate plane.

   ![Graphs of h(x) and g(x)](image)

   d. Use your graphs to estimate the solutions to the equation:

   \[ |x + 2| - 3 = -|x| + 4 \]

   Explain how you got your answer.

   \[ \text{Solution: } x \approx 2.5 \text{ or } x \approx -4.5 \]

   The solutions are the \( x \)-coordinates of the intersection points of the graphs of \( g \) and \( h \).

   e. Were your estimations you made in part (d) correct? If yes, how do you know? If not explain why not.

   \[ \begin{align*}
   \text{Let } x &= 2.5 \\
   \text{Is } |2.5 + 2| - 3 &= -|2.5| + 4 \text{ true?} \\
   \text{Yes, } 4.5 - 3 &= -2.5 + 4 \text{ is true.}
   \end{align*} \]

   \[ \begin{align*}
   \text{Let } x &= -4.5 \\
   \text{Is } |-4.5 + 2| - 3 &= -|4.5| + 4 \text{ true?} \\
   \text{Yes, } 2.5 - 3 &= -4.5 + 4 \text{ is true.}
   \end{align*} \]

   Yes, the estimates are correct. They each make the equation true.
2. Let \( f \) and \( g \) be the functions given by \( f(x) = x^2 \) and \( g(x) = x|x| \).

   a. Find \( f\left(\frac{1}{3}\right) \), \( g(4) \), and \( g(-\sqrt{3}) \).
      
      \[
      f\left(\frac{1}{3}\right) = \frac{1}{9}, \quad g(4) = 16, \quad g(-\sqrt{3}) = -3
      \]

   b. What is the domain of \( f \)?
      
      \( D: \) all real numbers.

   c. What is the range of \( g \)?
      
      \( R: \) all real numbers.

   d. Evaluate \( f(-67) + g(-67) \).
      
      \[
      (-67)^2 + -67|{-67}| = 0.
      \]

   e. Compare and contrast \( f \) and \( g \). How are they alike? How are they different?
      
      When \( x \) is positive, both functions give the same value. But when \( x \) is negative, 
      \( f \) gives the always positive value of \( x^2 \), whereas \( g \) gives a value that is the 
      opposite of what \( f \) gives.

   f. Is there a value of \( x \), such that \( f(x) + g(x) = -100 \) ? If so, find \( x \). If not, explain why no such 
      value exists.
      
      No, \( f \) and \( g \) are either both zero, giving a sum of zero, both positive, giving a 
      positive sum, or the opposite of each other, giving a sum of zero. So, there is 
      no way to get a negative sum.

   g. Is there a value of \( x \) such that \( f(x) + g(x) = 50 \) ? If so, find \( x \). If not, explain why no such value 
      exists.
      
      Yes, if \( x = 5 \), \( f(x) = g(x) = 25 \), thus \( f(x) + g(x) = 50 \).
3. A boy bought 6 guppies at the beginning of the month. One month later the number of guppies in his tank had doubled. His guppy population continued to grow in this same manner. His sister bought some tetrars at the same time. The table below shows the number of tetrars, \( t \), after \( n \) months have passed since they bought the fish.

<table>
<thead>
<tr>
<th>( n ), months</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t ), tetrars</td>
<td>8</td>
<td>16</td>
<td>24</td>
<td>32</td>
</tr>
</tbody>
</table>

a. Create a function \( g \) to model the growth of the boy’s guppy population, where \( g(n) \) is the number of guppies at the beginning of each month and \( n \) is the number of months that have passed since he bought the 6 guppies. What is a reasonable domain for \( g \) in this situation?

\[
g(n) = 6 \cdot 2^n \quad \text{Domain: } n \text{ is a whole number.}
\]

b. How many guppies will there be one year after he bought the 6 guppies?

\[
g(12) = 6 \cdot 2^{12} = 24,576 \text{ guppies}
\]

c. Create an equation that could be solved to determine how many months after he bought the guppies there will be 100 guppies.

\[
100 = 6 \cdot 2^n
\]

d. Use graphs or tables to approximate a solution to the equation from part (c). Explain how you arrived at your estimate.

At the end of 4 months, there are 96 guppies which is not quite 100, so during the 5th month, the guppy population reaches 100.

\[
n = 5 \text{ months}
\]
e. Create a function, \( t \), to model the growth of the sister’s tetra population, where \( t(n) \) is the number of tetras after \( n \) months have passed since she bought the tetras.
   \[
   t(n) = 8(n+1), \quad n \text{ is a whole number.}
   \]
   Or, \( t(n) = 8n + 8, \quad n \text{ is a whole number.} \)

f. Compare the growth of the sister’s tetra population to the growth of the guppy population. Include a comparison of the average rate of change for the functions that model each population’s growth over time.

The guppies’ population is increasing faster than the tetras’ population. Each month, the number of guppies doubles, while the number of tetra’s increases by 8. The rate of change for the tetras is constant, but the rate of change for the guppies is always increasing.

g. Use graphs to estimate the number of months that will have passed when the population of guppies and tetras will be the same.

The guppies and tetras populations will be the same, 24, after 2 months.
h. Use graphs or tables to explain why the guppy population will eventually exceed the tetra population even though there were more tets to start with.

The guppy population’s growth is exponential, and the tetra populations’ growth is linear. The graph in part (g) shows how the population of the guppies eventually overtakes the population of the tetras. The table below shows that by the end of the 3rd month, there are more guppies than tetras.

<table>
<thead>
<tr>
<th>n</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>g(n)</td>
<td>6</td>
<td>12</td>
<td>24</td>
<td>48</td>
<td>96</td>
<td>192</td>
</tr>
<tr>
<td>t(n)</td>
<td>8</td>
<td>16</td>
<td>24</td>
<td>32</td>
<td>40</td>
<td>48</td>
</tr>
</tbody>
</table>

i. Write the function \( g(n) \) in such a way that the percent increase in the number of guppies per month can be identified. Circle or underline the expression representing percent increase in number of guppies per month.

\[ g(n) = 6(200\%)^n \]
4. Regard the solid dark equilateral triangle as Figure 0. Then, the first figure in this sequence is the one composed of three dark triangles, the second figure is the one composed of nine dark triangles, and so on.

![Image of triangles](image)

**Figure 0**  **Figure 1**  **Figure 2**  **Figure 3**  **Figure 4**

<table>
<thead>
<tr>
<th>n (Figure Number)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>T (# of dark triangles)</td>
<td>1</td>
<td>3</td>
<td>9</td>
<td>27</td>
<td>81</td>
</tr>
</tbody>
</table>

a. How many dark triangles are in each figure? Make a table to show this data.

b. Describe in words how, given the number of dark triangles in a figure, to determine the number of dark triangles in the next figure.

> The number of triangles in a figure is 3 times the number of triangles in the previous figure.

c. Create a function that models this sequence. What is the domain of this function?

\[ T(n) = 3^n, \quad D: n \text{ is a whole number.} \]

d. Suppose the area of the solid dark triangle in Figure 0 is 1 square meter. The areas of one dark triangle in each figure form a sequence. Create an explicit formula that gives the area of just one of the dark triangles in the \( n \)th figure in the sequence.

\[
A(n) = \left(\frac{1}{2}\right)^n
\]

<table>
<thead>
<tr>
<th>Figure, n</th>
<th>Area of one dark triangle, A(n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1/4</td>
</tr>
<tr>
<td>2</td>
<td>1/16</td>
</tr>
<tr>
<td>3</td>
<td>1/64</td>
</tr>
</tbody>
</table>
e. The sum of the areas of all the dark triangles in Figure 0 is 1 m²; there is only one triangle in this case. The sum of the areas of all the dark triangles in Figure 1 is ¾ m². What is the sum of the areas of all the dark triangles in the \(n^{th}\) figure in the sequence? Is this total area increasing or decreasing as \(n\) increases?

<table>
<thead>
<tr>
<th>Figure</th>
<th>Area in m²</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>3/4</td>
</tr>
<tr>
<td>2</td>
<td>9/16</td>
</tr>
<tr>
<td>3</td>
<td>27/64</td>
</tr>
</tbody>
</table>

\[ T(n) = \left(\frac{3}{4}\right)^n \]

The total area is decreasing as \(n\) increases.

f. Let \(P(n)\) be the sum of the perimeters of the all dark triangles in the \(n^{th}\) figure in the sequence of figures. There is a real number \(k\) so that:

\[ P(n + 1) = kP(n) \]

is true for each positive whole number \(n\). What is the value of \(k\)?

Let \(x\) represent the number of meters long of one side of the triangle in Figure 0.

<table>
<thead>
<tr>
<th>Figure</th>
<th>(P(n))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3x</td>
</tr>
<tr>
<td>1</td>
<td>(3x + \frac{3}{2}x = \frac{9}{2}x)</td>
</tr>
<tr>
<td>2</td>
<td>(\frac{9}{2}x + \frac{9}{4}x = \frac{27}{4}x)</td>
</tr>
</tbody>
</table>

\(P\) is a geometric sequence and \(k\) is the ratio between any term and the previous term, so \(k = P(n+1)/P(n)\).

So, for example, for \(n = 0\),

\[ k = \frac{P(1)}{P(0)} = \frac{\frac{9}{2}x}{3x} = \frac{3}{2} \]

For \(n = 1\),

\[ k = \frac{P(2)}{P(1)} = \frac{\frac{27}{4}x}{\frac{9}{2}x} = \frac{3}{2} \]

\(k = \frac{3}{2}\)
5. The graph of a piecewise–defined function $f$ is shown to the right. The domain of $f$ is $-3 \leq x \leq 3$.

   a. Create an algebraic representation for $f$. Assume that the graph of $f$ is composed of straight line segments.

   $f(x) = \begin{cases} 
   x, & -3 \leq x < -1 \\
   -1, & -1 \leq x < 0 \\
   x - 1, & 0 \leq x < 2 \\
   -x + 3, & 2 \leq x \leq 3 
   \end{cases}$

   or $f(x) = \begin{cases} 
   x, & -3 \leq x < -1 \\
   -1, & -1 \leq x < 0 \\
   -|x - 2| + 1, & 0 \leq x \leq 3 
   \end{cases}$

   b. Sketch the graph of $y = 2f(x)$ and state the domain and range.

   Domain: $-3 \leq x \leq 3$

   Range: $-6 \leq y \leq 2$
c. Sketch the graph of \( y = f(2x) \) and state the domain and range.

\[
\text{Domain: } -1.5 \leq x \leq 1.5 \\
\text{Range: } -3 \leq y \leq 1
\]

![Graph of y = f(2x)](image)


d. How does the range of \( y = f(x) \) compare to the range of \( y = kf(x) \), where \( k > 1 \)?

Every value in the range of \( y = f(x) \) would be multiplied by \( k \). Since \( k > 1 \) we can represent this by multiplying the compound inequality that gives the range of \( y = f(x) \) by \( k \), giving \(-3k \leq y \leq k\).

e. How does the domain of \( y = f(x) \) compare to the domain of \( y = kf(x) \), where \( k > 1 \)?

Every value in the domain of \( y = f(x) \) would be divided by \( k \). Since \( k > 1 \) we can represent this by multiplying the compound inequality that gives the domain of \( y = f(x) \) by \( 1/k \), giving \(-3/k \leq x \leq 3/k\).