Grade 9 – Module 3 Topic C:
NOTE: Even though the topics in this curriculum were not necessarily designed to be considered as separate units, for these review purposes we are treating Topic C as a “unit” of study. Supporting materials for this module, which are related to Topic C, are also included. The End-of-Module Assessment included here, addresses content for all of the lessons in Module 3. In your review of Topic C as a unit, look at all assessment items for embedded evidence of understanding of the concepts addressed in Topic C, but look particularly look at items: 1, 2e, 5b, and 5c.
# Linear and Exponential Functions

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## Mid-Module Assessment and Rubric

Assessment 1 day, return 1 day, remediation or further applications 1 day.

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1 Each lesson is ONE day, and ONE day is considered a 45 minute period.
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Topics A through D (assessment 1 day, return 1 day, remediation or further applications 1 day)
Algebra I • Module 3

Linear and Exponential Functions

OVERVIEW

In earlier grades, students define, evaluate, and compare functions and use them to model relationships between quantities (8.F.A.1, 8.F.A.2, 8.F.A.3, 8.F.B.4, 8.F.B.5). In this module, students extend their study of functions to include function notation and the concepts of domain and range. They explore many examples of functions and their graphs, focusing on the contrast between linear and exponential functions. They interpret functions given graphically, numerically, symbolically, and verbally; translate between representations; and understand the limitations of various representations.

In Topic A, students explore arithmetic and geometric sequences as an introduction to the formal notation of functions (F-IF.A.1, F-IF.A.2). They interpret arithmetic sequences as linear functions with integer domains and geometric sequences as exponential functions with integer domains (F-IF.A.3, F-BF.A.1a). Students compare and contrast the rates of change of linear and exponential functions, looking for structure in each and distinguishing between additive and multiplicative change (F-IF.B.6, F-LE.A.1, F-LE.A.2, F-LE.A.3).

In Topic B, students connect their understanding of functions to their knowledge of graphing from Grade 8. They learn the formal definition of a function and how to recognize, evaluate, and interpret functions in abstract and contextual situations (F-IF.A.1, F-IF.A.2). Students examine the graphs of a variety of functions and learn to interpret those graphs using precise terminology to describe such key features as domain and range, intercepts, intervals where function is increasing or decreasing, and intervals where the function is positive or negative. (F-IF.A.1, F-IF.B.4, F-IF.B.5, F-IF.C.7a).

In Topic C, students extend their understanding of piecewise functions and their graphs including the absolute value and step functions. They learn a graphical approach to circumventing complex algebraic solutions to equations in one variable, seeing them as \( f(x) = g(x) \) and recognizing that the intersection of the graphs of \( f(x) \) and \( g(x) \) are solutions to the original equation (A-REI.D.11). Students use the absolute value function and other piecewise functions to investigate transformations of functions and draw formal conclusions about the effects of a transformation on the function’s graph (F-IF.C.7, F-BF.B.3).

Finally, in Topic D students apply and reinforce the concepts of the module as they examine and compare exponential, piecewise, and step functions in a real-world context (F-IF.C.9). They create equations and functions to model situations (A-CED.A.1, F-BF.A.1, F-LE.A.2), rewrite exponential expressions to reveal and relate elements of an expression to the context of the problem (A-SSE.B.3c, F-LE.B.5), and examine the key features of graphs of functions, relating those features to the context of the problem (F-IF.B.4, F-IF.B.6).

The Mid-Module Assessment follows Topic B. The End-of-Module Assessment follows Topic D.
Focus Standards for Mathematical Practice

MP.1  **Make sense of problems and persevere in solving them.** Students are presented with problems that require them to try special cases and simpler forms of the original problem to gain insight into the problem.

MP.2  **Reason abstractly and quantitatively.** Students analyze graphs of non-constant rate measurements and apply reason (from the shape of the graphs) to infer the quantities being displayed and consider possible units to represent those quantities.

MP.4  **Model with mathematics.** Students have numerous opportunities to solve problems that arise in everyday life, society, and the workplace (e.g., modeling bacteria growth, and understanding the federal progressive income tax system).

MP.7  **Look for and make use of structure.** Students reason with and analyze collections of equivalent expressions to see how they are linked through the properties of operations. They discern patterns in sequences of solving equation problems that reveal structures in the equations themselves. (e.g., $2x + 4 = 10, 2(x - 3) + 4 = 10, 2(3x - 4) + 4 = 10$)

MP.8  **Look for and express regularity in repeated reasoning.** After solving many linear equations in one variable (e.g., $3x + 5 = 8x - 17$), students look for general methods for solving a generic linear equation in one variable by replacing the numbers with letters (e.g., $ax + b = cx + d$). They pay close attention to calculations involving the properties of operations, properties of equality, and properties of inequalities, to find equivalent expressions and solve equations, while recognizing common ways to solve different types of equations.

Terminology

**New or Recently Introduced Terms**

- **Function** (A function is a correspondence between two sets, $X$ and $Y$, in which each element of $X$ is matched\(^{16}\) to one and only one element of $Y$. The set $X$ is called the *domain*; the set $Y$ is called the *range*.)
- **Domain** (Refer to the definition of *function*.)
- **Range** (Refer to the definition of *function*.)
- **Linear Function** (A linear function is a polynomial function of degree 1.)
- **Average Rate of Change** (Given a function $f$ whose domain includes the closed interval of real numbers $[a, b]$ and whose range is a subset of the real numbers, the average rate of change on the interval $[a, b]$ is $\frac{f(b) - f(a)}{b - a}$.)
- **Piecewise Linear Function** (Given non-overlapping intervals on the real number line, a (real) piecewise linear function is a function from the union of the intervals to the set of real numbers such that the function is defined by (possibly different) linear functions on each interval.)

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\(^{16}\)“Matched” can be replaced with “assigned” after students understand that each element of $x$ is matched to exactly one element of $y$. 
Familiar Terms and Symbols

- Numerical Symbol
- Variable Symbol
- Constant
- Numerical Expression
- Algebraic Expression
- Number Sentence
- Truth Values of a Number Sentence
- Equation
- Solution
- Solution Set
- Simple Expression
- Factored Expression
- Equivalent Expressions
- Polynomial Expression
- Equivalent Polynomial Expressions
- Monomial
- Coefficient of a Monomial
- Terms of a Polynomial

Suggested Tools and Representations

- Coordinate Plane
- Equations and Inequalities
- Graphing Calculator

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17 These are terms and symbols students have seen previously.
## Assessment Summary

<table>
<thead>
<tr>
<th>Assessment Type</th>
<th>Administered</th>
<th>Format</th>
<th>Standards Addressed</th>
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<td>After Topic B</td>
<td>Constructed response with rubric</td>
<td>F-IF.A.1, F-IF.A.2, F-IF.A.3, F-IF.B.4, F-IF.B.5, F-IF.B.6, F-IF.C.7a, F-BF.A.1a, F-LE.A.1, F-LE.A.2, F-LE.A.3</td>
</tr>
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</table>
Topic C: Transformations of Functions

A-REI.D.11, F-IF.C.7a, F-BF.B.3

Focus Standard: A-REI.D.11 Explain why the $x$-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.

F-IF.C.7a Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.*

a. Graph linear and quadratic functions and show intercepts, maxima, and minima.

F-BF.B.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k f(x)$, $f(kx)$, and $f(x + k)$ for specific values of $k$ (both positive and negative); find the value of $k$ given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

Instructional Days: 6

Lesson 15: Piecewise Functions
Lesson 16: Graphs Can Solve Equations Too
Lessons 17-20: Four Interesting Transformations of Functions

Lesson 15 of this Topic formalizes the study of piecewise functions that began in Module 1. The study of piecewise functions in this lesson includes step functions and the absolute value function. Piecewise functions work nicely in the remaining lessons of this topic beginning with Lesson 16, where students learn that an equation $f(x) = g(x)$, such as $|x - 3| + 1 = |2x - 4|$, can be solved by finding the intersection points of the graphs of $y = f(x)$ and $y = g(x)$. Students use technology in this lesson to create the graphs and observe their intersection points. Next, in Lessons 17-20 students use piecewise functions as they explore four transformations of functions: $f(x) + k$, $f(x + k)$, $kf(x)$, and $f(kx)$. 
Lesson 15: Piecewise Functions

Student Outcomes

- Students examine the features of piecewise functions including the absolute value function and step functions.
- Students understand that the graph of a function \( f \) is the graph of the equation \( y = f(x) \).

Lesson Notes

This lesson has two main purposes: The first is to continue the work from Lessons 11–13 regarding the interplay between graphs, equations and functions; the second is to expose students to piecewise functions in general and the absolute value and step functions, specifically. Lessons 12 and 13 established the meaning of the graph of a function and the graph of the equation \( y = f(x) \). This lesson continues to clarify that these two sets are one and the same. Students consider two important functions used in later lessons and classes: the absolute value function and the greatest integer function.

Classwork

Opening (2 minutes)

Recall that the absolute value of a number is the distance from 0 of a point on the number line. Because we are measuring distance, the absolute value of a non-zero number is always positive. For example, \(|-3| = 3\) because the point \(-3\), located 3 units to the left of 0 on the real number line is 3 units away from 0. Absolute value can also be used to define the distance between any two points on the real number line. For example, \(|5 - 8| = 3\) because there are 3 units between the numbers 5 and 8 on the real number line.

Opening Exercise (3 minutes) (optional)

Opening Exercise

For each real number \( a \), the absolute value of \( a \) is the distance between 0 and \( a \) on the number line and is denoted \(|a|\).

1. Solve each one variable equation.
   a. \( |x| = 6 \)
      \( (-6, 6) \)
   b. \( |x - 5| = 4 \)
      \( (9, 1) \)
   c. \( 2|x + 3| = -10 \)
      \( \text{No solution.} \)

2. Determine at least five solutions for each two-variable equation. Make sure some of the solutions include negative values for either \( x \) or \( y \).
   a. \( y = |x| \)
      \( ((-2, 2), (-1, -1), (0, 0), (1, 1), (2, 2)) \)
   b. \( y = |x - 5| \)
      \( ((-1, 6), (0, 5), (1, 4), (5, 0), (6, 1)) \)
   c. \( x = |y| \)
      \( ((1, 1), (1, -1), (0, 0), (2, 2), (2, -2)) \)

Scaffolding:

Much of this exploration relies on students accessing their knowledge from the beginning of this unit and from Module 1. Provide additional support as needed to reteach these ideas if students are struggling to work the exploration independently.
Exploratory Challenge 1 (15 minutes)

Have students work parts (a) – (d) in small groups. As you circulate, check to see that the groups are creating graphs. Remind them that the domain of the variables for these equations is all real numbers so their graphs should be continuous. Make sure groups are plotting (0,0) for parts (a) and (c) and (5,0) for part (b). After a few minutes, have different groups share their responses. Provide time for groups to revise their graphs as needed.

Part (d) offers an example of MP.6 as students must communicate their findings using precise language. A student example with particularly strong language may be highlighted for the benefit of the class.

Exploratory Challenge 1
For parts (a) – (c) create graphs of the solution set of each two-variable equation from Opening Exercise 2.

a. \( y = |x| \)

b. \( y = |x - 5| \)

c. \( x = |y| \)

d. Write a brief summary comparing and contrasting the three solution sets and their graphs.

The graphs of parts (a) and (b) are the same except that part (b) has point of the ‘vee’ (the vertex of angle) at \((5, 0)\) instead of \((0, 0)\). The other for part (c) looks like a 90° clockwise rotation of the graph from part (a) about the point \((0, 0)\). The points in the solution sets to parts (a) and (b) are a function but the points in the solution set for part (c) are not.
The next portion asks students to consider their work so far in Module 3. Part (h) makes the connection that the graph of the equation \( y = |x| \) and the graph of \( f \), where \( f(x) = |x| \) are identical. Question (j) is there to help students understand that the graph of \( y = f(x) \) and the graph of a two-variable equation (in \( x \) and \( y \)) are only identical if the elements of the equation’s solution set define a function in \( x \) to begin with.

For parts (e) – (j) consider the function \( f(x) = |x| \) where \( x \) can be any real number.

e. Explain the meaning of the function \( f \) in your own words.  
   *This function assigns every real number to its absolute value, which is the distance the point is located from 0 on the real number line. Each number and its opposite will have the same range element. The number 0 will be assigned to 0.*

f. State the domain and range of this function.
   *Domain: all real numbers. Range: all non-negative real numbers.*

g. Create a graph of the function \( f \). You might start by listing several ordered pairs that represent the corresponding domain and range elements.

![Graph of \( f(x) = |x| \)](image)

h. How does the graph of the absolute value function compare to the graph of \( y = |x| \)?
   *The two graphs are identical. They are identical because each ordered pair in the function would make the equation \( y = |x| \) a true number sentence if the domain value were substituted for \( x \) and the range value was substituted for \( y \). Therefore the graph of the function is the graph of the solution set of the equation.*

i. Define a function whose graph would be identical to the graph of \( y = |x - 5| \)?
   *Let \( g(x) = |x - 5| \) where \( x \) can be any real number.*

j. Could you define a function whose graph would be identical to the graph of \( x = |y| \)? Explain your reasoning.
   *No. The graph of \( x = |y| \) does not meet the definition of a graph of a function. If it were the graph of a function (say, the function \( h \)), it would be the set of ordered pairs \( \{(x, h(x)) \mid x \in D\} \), which means there would be only one \( y \)-value for each \( x \) in the domain \( D \). However, in the graph of \( x = |y| \) there is a number \( x \) (in fact, there are infinitely many \( x \)'s) associated with two different \( y \)-values: \((3,3)\) and \((3,-3)\) are both solutions to the equation \( x = |y| \).*
As you debrief questions (h)–(j) as a whole group, lead a discussion that includes a summary of the following information. When we create the graph of the solution set to a two-variable equation, we use essentially the same process as when we create the graph of \( y = f(x) \). We sift through all the \((x,y)\) pairs in the Cartesian plane and plot only those pairs that make a true number sentence. The difference between the two processes is that when we graph \( y = f(x) \), each \( x \) value in the domain of \( f \) will be paired with only one \( y \) value. When graphing a two-variable equation, there is no such restriction placed on the ordered pairs that return a true number sentence. The process of creating the graph of a function \( f \) yields the same results as graphing the solution set to the equation \( y = f(x) \) except we run through the set of domain values, determine the corresponding range value and then plot that ordered pair. Since each \( x \) in the domain is paired with exactly one \( y \) in the range, the resulting graphs will be the same. For this reason, we often use the variable symbol \( y \) and the function name \( f(x) \) interchangeably when we talk about the graph of a function or two-variable equation solved for \( y \). The caveat is that the two-variable equation must have a solution set where each \( x \) is paired with only one \( y \).

k. Let \( f_1(x) = -x \) for \( x < 0 \) and let \( f_2(x) = x \) for \( x \geq 0 \). Graph the functions \( f_1 \) and \( f_2 \) on the same Cartesian plane. How does the graph of these two functions compare to the graph in Exercise 7?

The graph of these two functions when graphed on the same Cartesian plane is identical to the graph of the absolute value function.

Close this portion of the lesson with the following definition of the absolute value function as a piecewise function.

Definition:

The absolute value function \( f \) is defined by setting \( f(x) = |x| \) for all real numbers. Another way to write \( f \) is as a piecewise linear function:

\[
f(x) = \begin{cases} 
-x & \text{if } x < 0 \\
\quad x & \text{if } x \geq 0
\end{cases}
\]
Example 1 (5 minutes)

This example shows students how to express a translation of the absolute value function as a piecewise function. Students create a graph of this function:

Example 1

Let \( g(x) = |x - 5| \). The graph of \( g \) is the same as the graph of the equation \( y = |x - 5| \) you drew in Exercise 3. Use the redrawn graph below to re-write the function \( g \) as a piecewise function.

Explain that we will need to derive the equations of both lines to write \( g \) as a piecewise function.

Label the graph of the linear function with negative slope by \( g_1 \) and the graph of the linear function with positive slope by \( g_2 \) as in the picture above.

Function \( g_1 \): Slope of \( g_1 \) is \(-1\) (why?), and the \( y \)-intercept is 5, therefore \( g_1(x) = -x + 5 \).

Function \( g_2 \): Slope of \( g_2 \) is \(1\) (why?), and the \( y \)-intercept is \(-5\) (why?), therefore \( g_2(x) = x - 5 \).

Writing \( g \) as a piecewise function is just a matter of collecting all of the different “pieces” and the intervals upon which they are defined:

\[
g(x) = \begin{cases} 
-x + 5 & \text{if } x < 5 \\
x - 5 & \text{if } x \geq 5 
\end{cases}
\]

- How does this graph compare to the graph of the translated absolute value function?
  - The graphs are congruent, but the graph of \( g \) has been translated to the right 5 units. (Using terms like “congruent” and “translated” reinforce concepts from 8th grade and prepare students for geometry.)

- How can you use your knowledge of the graph of \( f(x) = |x| \) to quickly determine the graph of \( g(x) = |x - 5| \)?
  - Watch where the vertex of the graph of \( f \) has been translated. In this case, \( g(x) = |x - 5| \) has translated the vertex point from \((0,0)\) to \((5,0)\). Then, graph a line with a slope of \(-1\) for the piece where \( x < 5 \) and a line with a slope of \(1\) for the piece, where \( x > 5 \).

- Can we interpret in words what this function does?
  - The range values are found by finding the distance between each domain element and the number 5 on the number line.
Exploratory Challenge 2 (8 minutes)

This exploration introduces the two types of step functions and a third function that is related to them: the floor function (also known as the greatest integer function), the ceiling function, and the sawtooth function. The notation that one often sees for the greatest integer function is \( f(x) = \lfloor x \rfloor \). Gauss first introduced the greatest integer function in the early 1800s. Later, Iverson defined the floor and ceiling functions and introduced the notation you see below in 1962. Both notations are used in mathematics. These functions are used in computer programming languages amongst other applications. Be sure to explain the notation.

Questions (b) and (c) will help students understand how the range values for each function are generated. In question (c), students will begin to understand that all real numbers in the interval have the same \( y \)-value. Clarify for students why the interval is closed at the left endpoint and open at the right endpoint. If students are struggling to create graphs, you may need to finish this exploration as a whole class. Before closing the lesson, make sure each student has a correct graph of the functions.

**Exploratory Challenge 2**

The floor of a real number \( x \), denoted by \( \lfloor x \rfloor \), is the largest integer not greater than \( x \). The ceiling of a real number \( x \), denoted by \( \lceil x \rceil \), is the smallest integer not less than \( x \). The sawtooth number of a positive number is the “fractional part” of the number that is to the right of its floor on the number line. In general, for a real number \( x \), the sawtooth number of \( x \) is the value of the expression \( x - \lfloor x \rfloor \). Each of these expressions can be thought of as functions with domain the set of real numbers.

**a.** Complete the following table to help you understand how these functions assign elements of the domain to elements of the range. The first and second rows have been done for you.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \text{floor}(x) = \lfloor x \rfloor )</th>
<th>( \text{ceiling}(x) = \lceil x \rceil )</th>
<th>( \text{sawtooth}(x) = x - \lfloor x \rfloor )</th>
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<tr>
<td>4.8</td>
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<td>(1)</td>
</tr>
<tr>
<td>(\pi)</td>
<td>3</td>
<td>4</td>
<td>(\pi - 3)</td>
</tr>
</tbody>
</table>

**b.** Create a graph of each function.

- \( \text{floor}(x) = \lfloor x \rfloor \)
- \( \text{ceiling}(x) = \lceil x \rceil \)
- \( \text{sawtooth}(x) = x - \lfloor x \rfloor \)
c. For the floor function, what would be the range value for all real numbers \( x \) on the interval \([0, 1)\)? The interval \([-2, -1)\)? The interval \([1, 2.5)\)?

- **Floor:** \([0)\), **Ceiling:** \([0, 1)\), **Sawtooth:** \([0, 1)\).
- **Floor:** \([1, 2)\), **Ceiling:** \([2)\), **Sawtooth:** \([0, 1)\).
- **Floor:** \((-2, -1)\), **Ceiling:** \([-2, -1)\), **Sawtooth:** \([0, 1)\).
- **Floor:** \([1, 2)\), **Ceiling:** \([2, 3)\), **Sawtooth:** \([0, 1)\).

Closing (2 minutes)

- You can use different expressions to define a function over different subsets of the domain. These are called piecewise functions. The absolute value function and step functions can be represented as piecewise functions.
- The graph of a function \( f \) and the graph of the equation \( y = f(x) \) are the same.

### Relevant Vocabulary

**PIECEWISE-LINEAR FUNCTION.** Given a number of non-overlapping intervals on the real number line, a **(real) piecewise-linear function** is a function from the union of the intervals to the set of real numbers such that the function is defined by (possibly different) linear functions on each interval.

**ABSOLUTE VALUE FUNCTION.** The absolute value of a number \( x \), denoted by \(|x|\), is the distance between 0 and \( x \) on the number line. The **absolute value function** is the piecewise-linear function such that for each real number \( x \), the value of the function is \(|x|\).

We often name the absolute value function by saying, “Let \( f(x) = |x| \) for all real numbers \( x \).”

**FLOOR FUNCTION.** The **floor** of a real number \( x \), denoted by \( \lfloor x \rfloor \), is the largest integer not greater than \( x \). The **floor function** is the piecewise-linear function such that for each real number \( x \), the value of the function is \( \lfloor x \rfloor \).

We often name the floor function by saying, “Let \( f(x) = \lfloor x \rfloor \) for all real numbers \( x \).”

**CEILING FUNCTION.** The **ceiling** of a real number \( x \), denoted by \( \lceil x \rceil \), is the smallest integer not less than \( x \). The **ceiling function** is the piecewise-linear function such that for each real number \( x \), the value of the function is \( \lceil x \rceil \).

We often name the ceiling function by saying, “Let \( f(x) = \lceil x \rceil \) for all real numbers \( x \).”

**SAWTOOTH FUNCTION.** The **sawtooth function** is the piecewise-linear function such that for each real number \( x \), the value of the function is given by the expression \( x - \lfloor x \rfloor \). The sawtooth function assigns to each positive number the part of the number (the non-integer part) that is to the right of the floor of the number on the number line. That is, if we let \( f(x) = x - \lfloor x \rfloor \) for all real numbers \( x \) then \( f\left(\frac{1}{3}\right) = \frac{1}{3}, f\left(\frac{1}{3}\right) = \frac{1}{3}, f(1000.02) = 0.02, f(-0.3) = 0.7, \) etc.

### Exit Ticket (5 minutes)
Lesson 15: Piecewise Functions

Exit Ticket

Each graph shown below represents the solution set to a two-variable equation.

Graph A    Graph B    Graph C

1. Which of these graphs could be represented by a function? Explain your reasoning.

2. For each one that can be represented by a function, define a piecewise function whose graph would be identical to the solution set shown.
Exit Ticket Sample Solutions

1. Which of these graphs could be represented by a function? Explain your reasoning.

*Graphs A and C could be represented by a function because each x in the domain is paired with exactly one y in the range.*

2. For each one that can be represented by a function, define a piecewise function whose graph would be identical to the solution set shown.

   **Graph A:**
   
   \[ f(x) = \begin{cases} 
   -x - 1, & x < -1 \\
   x + 1, & x \geq -1 
   \end{cases} \]

   **Graph C:**
   
   \[ f(x) = \begin{cases} 
   -2, & x < 0 \\
   0, & x = 0 \\
   2, & x > 0 
   \end{cases} \]

Problem Set Sample Solutions

These problems build student familiarity with piecewise functions and continue to reinforce the definition of function. The following solutions indicate an understanding of the objectives of this lesson:

1. Explain why the sawtooth function, \( \text{sawtooth}(x) = x - \lfloor x \rfloor \) for all real numbers \( x \), takes only the “fractional part” of a number when the number is positive.

   *If you subtract the integer part of a number from the number, only the “fractional part” will remain.*

2. Let \( g(x) = \lceil x \rceil - \lfloor x \rfloor \) where \( x \) can be any real number. In otherwords, \( g \) is the difference between the ceiling and floor functions. Express \( g \) as a piecewise function.

   \[ g(x) = \begin{cases} 
   0 & x \text{ is an integer} \\
   1 & x \text{ is not an integer} 
   \end{cases} \]

3. The Heaviside function is defined using the formula below.

   \[ H(x) = \begin{cases} 
   -1, & x < 0 \\
   0, & x = 0 \\
   1, & x > 0 
   \end{cases} \]

   Graph this function and state its domain and range.

   **Domain:** All real numbers.
   **Range:** \([-1, 0, 1] \).
4. The following piecewise function is an example of a step function.

\[ S(x) = \begin{cases} 
3 & -5 \leq x < -2 \\
1 & -2 \leq x < 3 \\
2 & 3 \leq x \leq 5 
\end{cases} \]

a. Graph this function and state the domain and range.
   - **Domain:** \([-5, 5]\)
   - **Range:** \(\{1, 2, 3\}\)

b. Why is this type of function called a step function?
   - The horizontal line segments step up and down like steps.

5. Let \(f(x) = \frac{|x|}{x}\) where \(x\) can be any real number except 0.

a. Why is the number 0 excluded from the domain of \(f\)?
   - If \(x = 0\) then the expression would not be defined.

b. What is the range of \(f\)?
   - \(\{-1, 1\}\)

c. Create a graph of \(f\).

d. Express \(f\) as a piecewise function.
   - \(f(x) = \begin{cases} 
-1 & x < 0 \\
1 & x > 0 
\end{cases} \)

e. What is the difference between this function and the Heaviside function?
   - The domain of the Heaviside function is all real numbers. The Heaviside function has a value of 0 when \(x = 0\). This function excludes the real number 0 from the domain.
6. Graph the following piecewise functions for the specified domain.
   
   a. \( f(x) = |x + 3| \) for \(-5 \leq x \leq 3\)

   ![Graph of f(x) = |x + 3| for -5 \leq x \leq 3]

   b. \( f(x) = |2x| \) for \(-3 \leq x \leq 3\)

   ![Graph of f(x) = |2x| for -3 \leq x \leq 3]

   c. \( f(x) = |2x - 5| \) for \(0 \leq x \leq 5\)

   ![Graph of f(x) = |2x - 5| for 0 \leq x \leq 5]
d. \( f(x) = |3x + 1| \) for \(-2 \leq x \leq 2\)

\[ \]

\[ \]

\[ \]

\[ \]

e. \( f(x) = |x| + x \) for \(-4 \leq x \leq 3\)

\[ \]

\[ \]

\[ \]

\[ \]

f. \( f(x) = \begin{cases} 
  x & \text{if } x \leq 0 \\
  x + 1 & \text{if } x > 0 
\end{cases} \)

\[ \]

\[ \]

\[ \]
7. Write a piecewise function for each graph below.

a. \( g(x) = \begin{cases} 2x + 3 & \text{if } x < -1 \\ 3 - x & \text{if } x \geq -1 \end{cases} \)

b. \( b(x) = \begin{cases} 2x + 4 & x < 0 \\ 4 & x \geq 0 \end{cases} \)
Lesson 15: Piecewise Functions

b. \( p(x) = \begin{cases} -3 & x \leq -2 \\ 1 & -2 \leq x < 2 \\ 2 & x > 2 \end{cases} \)

Graph of \( p \)

\[
\begin{array}{|c|c|}
\hline
x & y \\
\hline
-3 & -3 \\
1 & 1 \\
2 & 2 \\
\hline
\end{array}
\]


c. \( k(x) = \begin{cases} x + 3 & x < -1 \\ 2 & -2 \leq x < 2 \\ x + 1 & x > 2 \end{cases} \)

Graph of \( k \)

\[
\begin{array}{|c|c|}
\hline
x & y \\
\hline
-3 & -3 \\
2 & 2 \\
3 & 3 \\
\hline
\end{array}
\]

d. \( h(x) = \begin{cases} 4x - 3 & x < 0 \\ 2 & 0 \leq x \leq 2 \\ -2x + 8 & x > 2 \end{cases} \)

Graph of \( h \)

\[
\begin{array}{|c|c|}
\hline
x & y \\
\hline
-3 & -11 \\
2 & 2 \\
3 & 0 \\
\hline
\end{array}
\]
Lesson 16: Graphs Can Solve Equations Too

Student Outcomes

- Students discover that the multi-step and exact way of solving \(|2x - 5| = |3x + 1|\) using algebra can sometimes be avoided by recognizing that an equation of the form \(f(x) = g(x)\) can be solved visually by looking for the intersection points of the graphs of \(y = f(x)\) and \(y = g(x)\).

Lesson Notes

This lesson focuses on A.REI.11 which emphasizes that the \(x\)-coordinates of the intersection points of the graphs of two functions \(f\) and \(g\) are the solutions to the equation \(f(x) = g(x)\). This lesson ties work from Module 1 on solving systems of two-variable equations to work with functions and leads students to the understanding of what the solution set to a one-variable equation can be.

Classwork

Opening Exercises 1–3 (5 minutes)

In the opening, instruct students to solve for \(x\) in the equation by isolating the absolute value expression and separating the solution into two cases: one for the absolute value expression that represents distance from 0 in the positive direction of the number line, and one for the distance from 0 in the negative direction. In Exercise 2, we introduce the functions \(f\) and \(g\) somewhat artificially and consider the graphs \(y = f(x)\) and \(y = g(x)\). Students quickly recognize that this series of artificial moves actually has a solid purpose: to solve for \(x\) visually using the graphs of functions. After the labor required in Exercise 1, students should appreciate this clever way of solving for \(x\).

Opening Exercises 1–3

1. Solve for \(x\) in the following equation: \(|x + 2| - 3 = 0.5x + 1\)

\[
|x + 2| = 0.5x + 4
\]

\[
x + 2 = 0.5x + 4 \quad \text{or} \quad x + 2 = -(0.5x + 4)
\]

\[
0.5x = 2 \quad \text{or} \quad 1.5x = -6
\]

\[
x = 4 \quad \text{or} \quad x = -4
\]
2. Now let \( f(x) = |x + 2| - 3 \) and \( g(x) = 0.5x + 1 \). When does \( f(x) = g(x) \)? To answer this, first graph \( y = f(x) \) and \( y = g(x) \) on the same set of axes.

3. When does \( f(x) = g(x) \)? What is the visual significance of the points where \( f(x) = g(x) \)?

\[ f(x) = g(x) \text{ when } x = 4 \text{ and when } x = -4; (4, 3) \text{ and } (-4, -1). \text{ The points where } f(x) = g(x) \text{ are the intersections of the graphs of } f \text{ and } g. \]

4. Is each intersection point \((x, y)\) an element of the graph \( f \) and an element of the graph of \( g \)? In other words, do the functions \( f \) and \( g \) really have the same value when \( x = 4 \)? What about when \( x = -4 \)?

Yes. You can determine this by substituting \( x = -4 \) and \( x = 4 \) into both \( f \) and \( g \).

\[
\begin{align*}
-1 &= | -4 + 2 | - 3 \\
-1 &= 2 - 3 \\
-1 &= -1 \\
3 &= | 4 + 2 | - 3 \\
3 &= 6 - 3 \\
3 &= 3
\end{align*}
\]

\( f(x) \) and \( g(x) \) have the same value at each \( x \).

Be sure to review the solutions to these problems with the entire class before moving on. Before sharing as whole group, give students time to compare their answers with a partner.

Discussion (8 minutes)

Lead a discussion that ties together the work in the opening exercises. The idea here is to create an equation \( f(x) = g(x) \) and show that the \( x \)-coordinates of the intersection points are the solution set to this equation. This example will draw on MP.7 as students will need to look closely to determine the connection between the functions and equations involved.

First summarize what we know from the opening exercise. Ask students to discuss this with a partner and then call on a few people to share their thoughts. Make sure the following point is clear to the class:

- For functions \( f \) and \( g \), we have found special ordered pairs that (1) are points in the intersection of the graph of \( f \) and the graph of \( g \), that (2) are solutions to a system of equations given by the equations \( y = f(x) \) and \( y = g(x) \), and (3) where the \( x \)-value satisfies the equation \( f(x) = g(x) \).

Write this equation on the board. Ask students to use the solutions from the opening exercise to answer the following:

- Are the \( x \)-coordinates of the intersection points solutions to this equation?
  - Yes, when I substituted the \( x \)-coordinates into the equation I got a true number sentence.
Lesson 16

Graphs Can Solve Equations Too

Date: 9/12/13

Are the $y$-coordinates of the intersection points solutions to this equation?

- No, when I substituted the $y$-coordinates into the equation I got a false number sentence.

Do you think there are any other solutions to this equation? How could you be sure?

- I don’t think so, because each side of the equation is one of the functions shown in the graphs. The shape of the graph makes me think that there are no other intersection points. We could algebraically solve the equation to prove that these are the only solutions.

Give students (in groups of three or four) time to debate the next discussion question. Have different students share their thinking with the whole class.

- Is it always true that the $x$-coordinates of the intersection points of the graphs of two functions will be the solution set to the equation $f(x) = g(x)$?
  - Yes. To create the graphs of $f$ and $g$ we cycle through some of the domain values $x$ and plot the pairs $(x, f(x))$ and $(x, g(x))$. The points that these two functions have in common will have $x$-values that satisfy the equation $f(x) = g(x)$ because this equation asks us to find the domain elements $x$ that make the range elements $f(x)$ and $g(x)$ equal.

- What is the advantage of solving an equation graphically by finding intersection points in this manner?
  - It can be helpful when the equations are complicated or impossible to solve algebraically. It will also be useful when estimating solutions is enough to solve a problem. The graphically estimated solutions might give insight into ways to solve the equation algebraically.

Example 1 (8 minutes)

This example provides an opportunity to model explicitly how to use graphs of functions to solve an equation. As you work with students, guide them to label the graphs similarly to what is shown in the solutions below. This will reinforce proper vocabulary. In this guided example, students will complete the graphs of the functions and then fill in the blanks as you discuss as a whole class. This exercise should reinforce the previous discussion.

### Example 1

Solve this equation by graphing two functions on the same Cartesian plane:

$$|0.5x| - 5 = -|x - 3| + 4$$

Let $f(x) = |0.5x| - 5$ and let $g(x) = -|x - 3| + 4$ where $x$ can be any real number.

We are looking for values of $x$ at which the functions $f$ and $g$ have the same output value.

Therefore, we set $y = f(x)$ and $y = g(x)$ so we can plot the graphs on the same coordinate plane:

From the graph, we see that the two intersection points are

$(-4, -3)$ and $(8, -1)$.
The fact that the graphs of the functions meet at these two points means that when $x$ is ______ both $f(x)$ and $g(x)$ are ______, or when $x$ is ______ both $f(x)$ and $g(x)$ are ______.

$-4, -3, 8, -1$

Thus, the expressions $|0.5x| - 5$ and $-|x - 3| + 4$ are equal when $x = ______$ or when $x = ______$.

$-4, 8$

Therefore, the solution set to the original equation is _______.

$\{-4, 8\}$

After working with the class to use their knowledge from the previous lesson to create these graphs, lead a discussion that emphasizes the following:

- We are looking for values of $x$ where the values $f(x)$ and $g(x)$ are the same. In other words, we want to identify the points $(x, f(x))$ of the graph of $f$ and the points $(x, g(x))$ of the graph of $g$ that are the same. This will occur where the graphs of the two functions intersect.

- We must also convince ourselves that these are the only two solutions to this equation. Pose the question: How can we be certain that these two intersection points are the only two solutions to this equation? Give students time to discuss this with a partner or in a small group. Encourage them to reason from the graphs of the functions, rather than solving the equation.

  - For all $x < -4$ or $x > 8$, the differences in the $y$-values of two functions are always greater than zero. To see this, note that $f(x) - g(x) = |0.5x| - 5 - (-|x - 3| + 4) = |0.5x| + |x - 3| - 9$. The last expression is greater than zero when $|0.5x| + |x - 3| > 9$, which is certainly true for $x < -4$ or $x > 8$ (by inspection—there’s no need to solve this equation due to the graph). Hence, the only solutions can occur in the interval $-4 \leq x \leq 8$, of which there are two.

- If time permits, challenge students to experiment with sketching in the same Cartesian plane the graphs of two functions (each of which involve taking an absolute value) that intersect at 0 points, exactly 1 point, exactly two points, or an infinite number of points.

**Example 2 (10 minutes)**

This example requires graphing calculators or other graphing software that is capable of finding the intersection points of two graphs. As you work through this example, discuss and model how to:

- Enter functions into the graphing tool, graph them in an appropriate viewing window to see the intersection points, and use the features of the graphing technology to determine the coordinates of the intersection points.

- Show the difference between ‘tracing’ to the intersection point and using any built-in functions that determine the intersection point.

- Have students estimate the solutions from the graph before using the built-in features.

- Have students verify that the $x$-coordinates of the intersecting points are solutions to the equations.

- Have students sketch the graphs and label the coordinates of the intersection points on their handouts.

**Scaffolding:**

Refer to video lessons on the Internet for further examples and support for teaching this process using technology.
Example 2
Solve this equation graphically: \(-| x - 3.5 | + 4 = -0.25x - 1\)

a. Write the two functions represented by each side of the equation.

\[ f(x) = -| x - 3.5 | + 4 \quad \text{and} \quad g(x) = -0.25x - 1, \quad \text{where} \ x \ \text{can be any real number}. \]

b. Graph the functions in an appropriate viewing window.

![Graph of functions](image)

c. Determine the intersection points of the two functions.

\((-2, -1.5) \quad \text{and} \quad (6.8, 0.7)\)

d. Verify that the \(x\)-coordinates of the intersection points are solutions to the equation.

\[
\begin{align*}
\text{Let } x &= 2 \quad \text{then} \\
-| -2 - 3.5 | + 4 &= -0.25(-2) - 1 \\
-5.5 + 4 &= 0.5 - 1 \\
-0.5 &= -0.5
\end{align*}
\]

\[
\begin{align*}
\text{Let } x &= 6.75 \quad \text{then} \\
-| 6.8 - 3.5 | + 4 &= -0.25(6.8) - 1 \\
-3.3 + 4 &= 1.7 - 1 \\
0.7 &= 0.7
\end{align*}
\]

Exercises 1–5 (8 minutes)

Students practice using graphs of functions to solve equations. Students should work through these exercises in small groups and discuss their solutions as they work. Circulate among groups providing assistance as needed.
Exercises 1–5

Use graphs to find approximate values of the solution set for each equation. Use technology to support your work. Explain how each of your solutions relates to the graph. Check your solutions using the equation.

1. \( 3 - 2x = |x - 5| \)
   \[ x = -2, \text{ the intersection point is } (-2, 7). \]

2. \( 2(1.5)^x = 2 + 1.5x \)
   \[ \text{First solution is } x = 0, \text{ from the point } (0, 2); \]
   \[ \text{Second solution answers will vary, } x \text{ is about } 2.7 \text{ or } 2.8, \]
   \[ \text{based on the actual intersection point of } (2.776, 6.164) \]

3. The graphs of the functions \( f \) and \( g \) are shown.
   a. Use the graph to approximate the solution(s) to the equation \( f(x) = g(x) \).
      
      Based on the graphs, the approximate solutions are \((-0.7, 2)\).
   b. Let \( f(x) = x^2 \) and let \( g(x) = 2^x \). Find all solutions to the equation \( f(x) = g(x) \). Verify any exact solutions that you determine using the definitions of \( f \) and \( g \). Explain how you arrived at your solutions.
      
      By guessing and checking, \( x = 4 \) is also a solution of the equation because \( f(4) = 16 \) and \( g(4) = 16 \). Since the graph of the exponential function is increasing and increases more rapidly than the squaring function, there will only be 3 solutions to this equation. The exact solutions are \( x = 2 \) and \( x = 4 \) and an approximate solution is \( x = -0.7 \).

4. The graphs of \( f \), a function that involves taking an absolute value, and \( g \), a linear function, are shown to the right. Both functions are defined over all real values for \( x \). Tami concluded that the equation \( f(x) = g(x) \) has no solution.
   Do you agree or disagree? Explain your reasoning.
   
   I disagree with Tami because we cannot see enough of this graph. The graph of the function shown to the left has a slope of 5. The graph of the function shown to the right has a slope greater than 5. Therefore, these two functions will intersect somewhere in the first quadrant. We would have to ‘zoom out’ to see the intersection point.
5. The graphs of $f$ (a function that involves taking the absolute value) and $g$ (an exponential function) are shown below. Sharon said the solution set to the equation $f(x) = g(x)$ is exactly $\{-7, 5\}$.

Do you agree or disagree with Sharon? Explain your reasoning.

I disagree with Sharon. We could say that the solution set is approximately $\{-7, 5\}$ but without having the actual equation or formulas for the two functions we cannot be sure the $x$-values of the intersection points are exactly $-7$ and $5$.

Closing (2 minutes)

In the last two exercises, students reflect on the limitations of solving an equation graphically. Debrief these exercises as a whole class and encourage different groups to present their reasoning to the entire class. Clarify any misconceptions before moving on, and give students time to revise their work. In Exercise 4, it is clear that there is an intersection point that is not visible in the viewing window provided. In Exercise 5, the intersection points would need to be estimated. If we do not have the exact algebraic solutions of the equation, then we can only estimate the solution set using graphs.

Exit Ticket (5 minutes)
Lesson 16: Graphs Can Solve Equations Too

Exit Ticket

1. How do intersection points of the graphs of two functions $f$ and $g$ relate to the solution to an equation in the form $f(x) = g(x)$?

2. What are some benefits of solving equations graphically? What are some limitations?
Exit Ticket Sample Solutions

1. How do intersection points of the graphs of two functions \( f \) and \( g \) relate to the solution to an equation in the form \( f(x) = g(x) \)?

The \( x \)-coordinates of the intersection points of the graphs of two functions are the solutions of the equation.

2. What are some benefits of solving equations graphically? What are some limitations?

**Benefits:** Solving equations graphically can be helpful when you don’t know how to solve the equation algebraically. It can also save you some time if you have technology available. This method can only provide approximate solutions, which may be all you need. Or the approximate solutions may give you insight into how to solve the equation algebraically.

**Limitations:** You cannot be sure you have found all the solutions to an equation unless you can reason about the graphs of the functions themselves and convince yourself that no other intersection points are possible. The solutions found graphically rely on eyeballing. There is no guarantee that they are exact solutions; sometimes they are, other times they are just decent approximations.

Problem Set Sample Solutions

1. Solve the following equations graphically. Verify the solution set using the original equations.

   a. \( 2x - 4 = \sqrt{x} + 5 \)
      
      Approximately 3.4538

   b. \( |x| = x^2 \)
      
      \((-1, 0, 1)\)

   c. \( x + 2 = x^3 - 2x - 4 \)
      
      Approximately 2.3553

   d. \( |3x - 4| = 5 - |x - 2| \)
      
      \(\{0.25, 2.75\}\)

   e. \( 0.5x^2 - 4 = 3x + 1 \)
      
      Approximately 3.0467

   f. \( 6\left(\frac{1}{2}\right)^{5x} = 10 - 6x \)
      
      Approximately \(-0.1765 and 1.6636\)
In each exercise, the graphs of the functions $f$ and $g$ are shown on the same Cartesian plane. Estimate the solution set to the equation $f(x) = g(x)$. Assume that the graphs of the two functions only intersect at the points shown on the graph.

2. \{3, 9\}

3. \{-3, 1\}

4. \{1\}

5. \{1, 2, 6\}
6. The graph below shows Glenn’s distance from home as he rode his bicycle to school, which is just down his street. His next-door neighbor Pablo, who lives 100 m closer to the school, leaves his house at the same time as Glenn. He walks at a constant velocity and they both arrive at school at the same time.

![Graph showing Glenn's and Pablo's distances from home over time]

a. Graph a linear function that represents Pablo’s distance from Glenn’s home as a function of time.

b. Estimate when the two boys pass each other.

*They cross paths at about 2 minutes and 5 minutes. I can tell that by finding the x-coordinates of the intersection points of the graphs of the functions.*

c. Write piecewise-linear functions to represent each boy’s distance and use them to verify your answer to part (b).

\[
P(t) = 100 + 37.5t
\]

\[
G(t) = \begin{cases} 
250/3t & 0 \leq t \leq 3 \\
200 + 50/3t & 3 < t \leq 6 \\
50t & 6 < t \leq 8
\end{cases}
\]

At about 2 minutes: \(100 + \frac{75}{2}t = \frac{250}{3}t\) or \(600 + 225t = 500t\) or \(275t = 600\) or \(t = \frac{24}{11}\) min.

At about 5 minutes: \(100 + \frac{75}{2}t = 200 + \frac{50}{3}t\) or \(225t = 600 + 100t\) or \(125t = 600\) or \(t = \frac{24}{5} = 4.8\) min.
Lesson 17: Four Interesting Transformations of Functions

Student Outcomes

- Students examine that a vertical translation of the graph of \( y = f(x) \) corresponds to changing the equation from \( y = f(x) \) to \( y = f(x) + k \).
- Students examine that a vertical scaling of the graph of \( y = f(x) \) corresponds to changing the equation from \( y = f(x) \) to \( y = kf(x) \).

Lesson Notes

Students enter Algebra 1 having experience with transforming lines, rays, triangles, etc., using translations, rotations, reflections, and dilations from Grade 8 Modules 2 and 3. Thus, it is natural to begin a discussion of transformations of functions by transforming graphs of functions—the graph of a function, \( f: \mathbb{R} \rightarrow \mathbb{R} \), is just another geometric figure in the (Cartesian) plane. Students use language such as, “a translation 2 units to the left,” or, “a vertical stretch by a scale factor of 3,” to describe how the original graph of the function is transformed into the new graph geometrically.

As students apply their Grade 8 geometry skills to the graph of the equation \( y = f(x) \), they realize that the translation of the graph to the right by 4 units is given by the graph of the equation \( y = f(x - 4) \). This recognition, in turn, leads to the idea of a transformation of a function. (i.e., a new function such that the graph of it is the transformation of the original graph of \( y = f(x) \).) In the example described, it is the function given by \( g(x) = f(x - 4) \) for any real number \( x \) such that \( x - 4 \) is in the domain of \( f \).

Since the transformation of the function is itself another function (and not a graph), we must use function language to describe the transformation. A function \( f \) cannot be translated up, down, right or left (even though its graph can). Rather, students can use function language such as: “For the same inputs, the values of the transformed function are two times as large as the values of the original function.”

These lessons encourage fluidity in both the language associated with transformations of graphs, and the language associated with transformations of functions. While a formal definition for the transformation of a function is not included, teachers are encouraged use language precisely as students work to develop the notion of transformation of a function and relate it to their understanding of transformations of graphical objects.

In the exploratory challenge, you may highlight MP.3 by asking students to make a conjecture about the effect of \( k \). This challenge also calls on students to employ MP.8, as they will generalize the effect of \( k \) through repeated graphing.
Classwork

Exploratory Challenge 1/Example 1 (12 minutes)

Let \( f(x) = |x| \) for all real numbers \( x \). Students explore the effect on the graph of \( y = f(x) \) by changing the equation \( y = f(x) \) to \( y = f(x) + k \) for given values of \( k \).

Exploratory Challenge 1/Example 1

Let \( f(x) = |x|, g(x) = f(x) - 3, h(x) = f(x) + 2 \) for any real number \( x \).

1. Write an explicit formula for \( g(x) \) in terms of \( |x| \) (i.e., without using \( f(x) \) notation):
   \[ g(x) = |x| - 3 \]

2. Write an explicit formula for \( h(x) \) in terms of \( |x| \) (i.e., without using \( f(x) \) notation):
   \[ h(x) = |x| + 2 \]

3. Complete the table of values for these functions.

   \[
   \begin{array}{|c|c|c|c|}
   \hline
   x & f(x) = |x| & g(x) = f(x) - 3 & h(x) = f(x) + 2 \\
   \hline
   -3 & 3 & 0 & 5 \\
   -2 & 2 & -1 & 4 \\
   -1 & 1 & -2 & 3 \\
   0 & 0 & -3 & 2 \\
   1 & 1 & -2 & 3 \\
   2 & 2 & -1 & 4 \\
   3 & 3 & 0 & 5 \\
   \hline
   \end{array}
   \]

4. Graph all three equations: \( y = f(x), y = f(x) - 3, \) and \( y = f(x) + 2 \).
5. What is the relationship between the graph of \( y = f(x) \) and the graph of \( y = f(x) + k \)?

For values of \( k \), where \( k > 0 \), for every point \((x, f(x))\) that satisfies the equation \( y = f(x) \), there is a corresponding point \((x, f(x) + k)\) on the graph, located \( k \) units above \((x, f(x))\) that satisfies the equation \( y = f(x) + k \). The graph of \( y = f(x) + k \) is the vertical translation of the graph of \( y = f(x) \) by \( k \) units upward.

For values of \( k \), where \( k < 0 \), for every point \((x, f(x))\) that satisfies the equation \( y = f(x) \), there is a corresponding point \((x, f(x) + k)\) on the graph, located \( k \) units below \((x, f(x))\) that satisfies the equation \( y = f(x) + k \). The graph of \( y = f(x) + k \) is the vertical translation of the graph of \( y = f(x) \) by \( k \) units downward.

The use of transformation language like “vertical translation” is purposeful. The Common Core State Standards require students to spend a lot of time talking about translations, rotations, reflections, and dilations in Grade 8. They will spend more time in Grade 10. Reinforcing this vocabulary will help to link these grades together.

6. How do the values of \( g \) and \( h \) relate to the values of \( f \)?

For each \( x \) in the domain of \( f \) and \( g \), the value of \( g(x) \) is 3 less than the value of \( f(x) \). For each \( x \) in the domain of \( f \) and \( h \), the value of \( h(x) \) is 2 more than the value of \( f(x) \).

**Discussion (3 minutes)**

Students should finish Example 1 with the understanding that the graph of a function \( g \) found by adding a number to another function, as in \( g(x) = f(x) + k \), is the translation of the graph of the function \( f \) vertically by \( k \) units (positively or negatively depending on the sign of \( k \)).

**Exploratory Challenge 2/Example 2 (12 minutes)**

Let \( f(x) = |x| \) for any real number \( x \). Students explore the effect on the graph of \( y = f(x) \) by changing the equation \( y = f(x) \) to \( y = kf(x) \) for given values of \( k \).

**Exploratory Challenge 2/Example 2**

1. Let \( f(x) = |x|, g(x) = 2f(x), h(x) = \frac{1}{2}f(x) \) for any real number \( x \). Write a formula for \( g(x) \) in terms of \( |x| \) (i.e., without using \( f(x) \) notation):
   \[
   g(x) = 2|x|
   \]

2. Write a formula for \( h(x) \) in terms of \( |x| \) (i.e., without using \( f(x) \) notation):
   \[
   h(x) = \frac{1}{2} |x|
   \]
3. Complete the table of values for these functions.

| $x$ | $f(x) = |x|$ | $g(x) = 2f(x)$ | $h(x) = \frac{1}{2}f(x)$ |
|-----|-------------|---------------|------------------|
| −3  | 3           | 6             | 1.5              |
| −2  | 2           | 4             | 1                |
| −1  | 1           | 2             | 0.5              |
| 0   | 0           | 0             | 0                |
| 1   | 1           | 2             | 0.5              |
| 2   | 2           | 4             | 1                |
| 3   | 3           | 6             | 1.5              |

4. Graph all three equations: $y = f(x)$, $y = 2f(x)$, and $y = \frac{1}{2}f(x)$.

Let $p(x) = −|x|$, $q(x) = −2f(x)$, $r(x) = −\frac{1}{2}f(x)$ for any real number $x$.

5. Write the formula for $q(x)$ in terms of $|x|$ (i.e., without using $f(x)$ notation):
   $q(x) = −2|x|$

6. Write the formula for $r(x)$ in terms of $|x|$ (i.e., without using $f(x)$ notation):
   $r(x) = −\frac{1}{2}|x|$
7. Complete the table of values for the functions \( p(x) = -|x|, \) \( q(x) = -2f(x), \) \( r(x) = -\frac{1}{2}f(x). \)

| \( x \) | \( p(x) = -|x| \) | \( q = -2f(x) \) | \( r(x) = -\frac{1}{2}f(x) \) |
|---|---|---|---|
| -3 | -3 | -6 | -1.5 |
| -2 | -2 | -4 | -1 |
| -1 | -1 | -2 | -0.5 |
| 0 | 0 | 0 | 0 |
| 1 | -1 | -2 | -0.5 |
| 2 | -2 | -4 | -1 |
| 3 | -3 | -6 | -1.5 |

8. Graph all three functions on the same graph as \( y = p(x), y = q(x), \) and \( y = r(x). \)

9. How is the graph of \( y = f(x) \) related to the graph of \( y = kf(x) \) when \( k > 1? \)

The graph of \( y = kf(x) \) for \( k > 1 \) contains points \((x, ky)\) which are related to points \((x, y)\) in the graph of \( y = f(x). \) The number \( ky \) is a multiple of \( y; \) each \( y\)-value of \( y = g(x) \) is \( k \) times the \( y\)-value of \( y = f(x). \) The graph of \( y = kf(x) \) is a vertical scaling that appears to stretch the graph of \( y = f(x) \) vertically by a factor of \( k. \)
10. How is the graph of \( y = f(x) \) related to the graph of \( y = kf(x) \) when \( 0 < k < 1 \)?

The graph of \( y = kf(x) \) for \( 0 < k < 1 \) contains points \( (x, ky) \) which are related to points \( (x, y) \) in the graph of \( y = f(x) \). The number \( ky \) is a fraction of \( y \); each \( y \)-value of \( y = g(x) \) is \( k \) times the \( y \)-value of \( y = f(x) \). The graph of \( y = kf(x) \) is a vertical scaling that appears to shrink the graph of \( y = f(x) \) vertically by a factor of \( k \).

11. How do the values of functions \( p, q, \) and \( r \) relate to the values of functions \( f, g, \) and \( h \), respectively? What transformation of the graphs of \( f, g, \) and \( h \) represents this relationship?

Each function is the opposite of the corresponding function. The result is that each \( y \)-value of any point on the graph of \( y = p(x) \), \( y = q(x) \), and \( y = r(x) \) are the opposite of the \( y \)-value of the graphs of the equations \( y = f(x) \), \( y = g(x) \), and \( y = h(x) \). Each graph is a reflection of the corresponding graph over the \( x \)-axis.

Discussion (3 minutes)

Students should finish Example 2 with the understanding that a number, a scale factor, multiplied to a function vertically scales the original graph. For a vertical scale factor of \( k > 1 \), the graph is a vertical stretch of the original graph; for a vertical scale factor of \( k \) where \( 0 < k < 1 \), the graph is a vertical shrink of the original graph. For a vertical scale factor of \( k \) where \( -1 < k < 0 \), the graph of the function is a reflection across the \( x \)-axis of the graph when \( 0 < k < 1 \). Similarly, for a vertical scale factor of \( k < -1 \), the graph is the reflection across the \( x \)-axis of the graph when \( k > 1 \).

Scaffolding:
Gives guidance specific to this proportion of the lesson for addressing needs of diverse learners. The type of diverse learner should be specified (i.e., advanced learner, etc.)

Exercises (8 minutes)

Students complete exercises independently; then compare/discuss with partner or small group. Circulate to ensure that students grasp the effects of the given transformations.

**Exercises**

1. Make up your own function \( f \) by drawing the graph of it on the Cartesian plane below. Label it as the graph of the equation, \( y = f(x) \). If \( b(x) = f(x) - 4 \) and \( c(x) = \frac{1}{3} f(x) \) for every real number \( x \), graph the equations \( y = b(x) \) and \( y = c(x) \) on the same Cartesian plane.

Answers will vary. Look for and encourage students to create interesting graphs for their function \( f \). (Functions DO NOT have to be defined by algebraic expressions—any graph that satisfies the definition of a function will do.) One such option is using \( f(x) = |x| \), as shown in the example below.
If time permits, have students present their graphs to the class and explain how they found the graphs of $y = b(x)$ and $y = c(x)$. Pay close attention to how students explain how they found the graph of $y = c(x)$. Many might actually describe a horizontal scaling (or some other transformation that takes each point $(x, y)$ of the graph to another point that does not have the same $x$-coordinate). Stress that multiplying the function $f$ by $k$ only scales the $y$-coordinate and leaves the $x$-coordinate alone.

Closing (3 minutes)

Point out that there is nothing special about using the function $f(x) = |x|$ as we did in this lesson. These transformations hold in general:

- Discuss how the graph of $y = f(x)$ can be vertically translated by positive or negative $k$. Draw a graph of a made up function on the board, labeled by $y = f(x)$, and show how to translate it up or down by $k$ using the equation $y = f(x) + k$.
- Discuss how the graph of $y = f(x)$ can be vertically scaled by $k$ for $0 < k < 1$, $k > 1$, $-1 < k < 0$, $k < -1$. Use the graph of $y = f(x)$ to show how to vertically scale (i.e., vertically stretch or shrink) by $k$ units using the equation $y = kf(x)$.

Exit Ticket (5 minutes)
Lesson 17: Four Interesting Transformations of Functions

Exit Ticket

Let \( p(x) = |x| \) for every real number \( x \). The graph of \( y = p(x) \) is shown below.

1. Let \( q(x) = -\frac{1}{2}|x| \) for every real number \( x \). Describe how to obtain the graph of \( y = q(x) \) from the graph of \( y = p(x) \). Sketch the graph of \( y = q(x) \) on the same set of axes as the graph of \( y = p(x) \).

2. Let \( r(x) = |x| - 1 \) for every real number \( x \). Describe how to obtain the graph of \( y = r(x) \) from the graph of \( y = p(x) \). Sketch the graph of \( y = r(x) \) on the same set of axes as the graphs of \( y = p(x) \) and \( y = q(x) \).
Exit Ticket Sample Solutions

Let \( p(x) = |x| \) for every real number \( x \). The graph of \( y = p(x) \) is shown below.

1. Let \( q(x) = -\frac{1}{2}|x| \) for every real number \( x \). Describe how to obtain the graph of \( y = q(x) \) from the graph of \( y = p(x) \). Sketch the graph of \( y = q(x) \) on the same set of axes as the graph of \( y = p(x) \).

   Reflect and vertically scale the graph of \( y = p(x) \) by plotting \( \left( x, -\frac{1}{2}y \right) \) for each point \( (x, y) \) in the graph of \( y = p(x) \). See the graph of \( q(x) \) below.

2. Let \( r(x) = |x| - 1 \) for every real number \( x \). Describe how to obtain the graph of \( y = r(x) \) from the graph of \( y = p(x) \). Sketch the graph of \( y = r(x) \) on the same set of axes as the graphs of \( y = p(x) \) and \( y = q(x) \).

   Translate the graph of \( y = p(x) \) vertically down 1 unit. See the graph of \( y = r(x) \) below.

Problem Set Sample Solutions

Let \( f(x) = |x| \) for every real number \( x \). The graph of \( y = f(x) \) is shown below. Describe how the graph for each function below is a transformation of the graph of \( y = f(x) \). Then use this same set of axes to graph each function for problems 1 – 5. Be sure to label each function on your graph (by \( y = a(x) \), \( y = b(x) \), etc.).

1. \( a(x) = |x| + \frac{3}{2} \)

   Translate the graph of \( y = f(x) \) up 1.5 units.

2. \( b(x) = -|x| \)

   Reflect \( y = f(x) \) across the \( x \)-axis.
3. \( c(x) = 2|x| \)
   
   Vertically scale/stretch the graph of \( y = f(x) \) by doubling the output values for every input.

4. \( d(x) = \frac{1}{3}|x| \)
   
   Vertically scale/shrink the graph of \( y = f(x) \) by dividing the output values by 3 for every input.

5. \( e(x) = |x| - 3 \)
   
   Translate the graph of \( y = f(x) \) down 3 units.

6. Let \( r(x) = |x| \) and \( t(x) = -2|x| + 1 \) for every real number \( x \). The graph of \( y = r(x) \) is shown below. Complete the table below to generate output values for the function \( t \); then graph the equation \( y = t(x) \) on the same set of axes as the graph of \( y = r(x) \).

| \( x \) | \( r(x) = |x| \) | \( t(x) = -2|x| + 1 \) |
|---|---|---|
| -2 | 2 | -3 |
| -1 | 1 | -1 |
| 0 | 0 | 1 |
| 1 | 1 | -1 |
| 2 | 2 | -3 |
7. Let $f(x) = |x|$ for every real number $x$. Let $m$ and $n$ be functions found by transforming the graph of $y = f(x)$. Use the graphs of $y = f(x), y = m(x)$ and $y = n(x)$ below to write the functions $m$ and $n$ in terms of the function $f$. (Hint: what is the $k$?)

- $m(x) = 2f(x)$.
- $n(x) = f(x) + 2.$
Lesson 18: Four Interesting Transformations of Functions

Student Outcomes
- Students examine that a horizontal translation of the graph of \( y = f(x) \) corresponds to changing the equation from \( y = f(x) \) to \( y = f(x + k) \).

Lesson Notes
In Lesson 18, students examine horizontal translations (shifts) in the graph of a function and how they are represented in the equation of the function. Students will contrast the horizontal shift to the vertical shift covered in Lesson 17. They should be able to describe the transformations of the graph associated with the transformation of the function, as well as write the equation of a graph based on the translations (shifts) or vertical scalings (stretches) of another graph whose equation is known.

Classwork
Example 1 (8 minutes)
Students explore that a horizontal translation of the graph of \( y = f(x) \) corresponds to changing the equation from \( y = f(x) \) to \( y = f(x + k) \) for given values of \( k \). As an example of MP.3, consider asking students to make a conjecture about how they believe this placement of \( k \) will affect the graph.

Example 1
Let \( f(x) = |x|, \ g(x) = f(x - 3), \ h(x) = f(x + 2) \) where \( x \) can be any real number.

a. Write the formula for \( g(x) \) in terms of \( |x| \) (i.e., without using \( f(x) \) notation):
   \[ g(x) = |x - 3| \]

b. Write the formula for \( h(x) \) in terms of \( |x| \) (i.e., without using \( f(x) \) notation):
   \[ h(x) = |x + 2| \]

c. Complete the table of values for these functions.

| \( x \) | \( f(x) = |x| \) | \( g(x) = f(x - 3) \) | \( h(x) = f(x + 2) \) |
|-------|----------------|----------------|----------------|
| -3    | 3              | 6              | 1              |
| -2    | 2              | 5              | 0              |
| -1    | 1              | 4              | 1              |
| 0     | 0              | 3              | 2              |
| 1     | 1              | 2              | 3              |
| 2     | 2              | 1              | 4              |
| 3     | 3              | 3              | 5              |
Lesson 18

Four Interesting Transformations of Functions

Date: 9/12/13

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Scaffolding:
Gives guidance specific to this proportion of the lesson for addressing needs of diverse learners. The type of diverse learner should be specified (i.e., advanced learner, etc.)

Discussion (5 minutes)

Students should finish Example 1 with the understanding that the graph of a function $g$ found by subtracting a number $k$ to the input of another function, as in $g(x) = f(x - k)$, is a translation of the graph of the function $f$ horizontally by $k$ units (positively or negatively, depending on the sign of $k$).

- If we replace 3 by a number $k$ in $g(x) = f(x - 3)$ as in Example 1 to get $g(x) = f(x - k)$, explain how to translate the graph of $f$ to the graph of $g$ in terms of $k$.
  - If $k > 0$, then the graph of $f$ is translated to the right by $|k|$ units.
  - If $k < 0$, then the graph of $f$ is translated to the left by $|k|$ units.

- How does the graph of $y = f(x)$ relate to the graph of $y = f(x - 3)$?
  
The graph of $f(x - 3)$ is the graph of $f(x)$ translated horizontally to the right 3 units.

- How does the graph of $y = f(x)$ relate to the graph of $y = f(x + 2)$?
  
The graph of $f(x + 2)$ is the graph of $f(x)$ translated horizontally to the left 2 units.

- How does the graph of $y = |x| - 3$ and the graph of $y = |x - 3|$ relate differently to the graph of $y = |x|$?
  
The graph of $y = |x| - 3$ translates the graph of $y = |x|$ down 3 units whereas the graph of $y = |x - 3|$ translates the graph of $y = |x|$ to the right 3 units.

- How do the values of $g$ and $h$ relate to the values of $f$?
  
The input value for $g$ has to be 3 more than the input value for $f$ to get the same output values. The input value for $h$ has to be two more than the input value for $f$ to get the same output values.

- Graph all three equations: $y = f(x), y = f(x - 3)$, and $y = f(x + 2)$.

- The graph of $y = f(x - 3)$ is the graph of $f(x)$ translated horizontally to the right 3 units.

- The graph of $y = f(x + 2)$ is the graph of $f(x)$ translated horizontally to the left 2 units.

- The input value for $g$ has to be 3 more than the input value for $f$ to get the same output values. The input value for $h$ has to be two more than the input value for $f$ to get the same output values.
In general, for any \( k \), the graph of \( f \) is translated horizontally by \( k \) units (where \( k > 0 \) corresponds to a translation to the right and \( k < 0 \) corresponds to a translation to the left).

- How does your answer for \( k < 0 \) make sense for \( h(x) = f(x + 2) \)?
  - We can rewrite \( h(x) = f(x + 2) \) as \( h(x) = f(x - (-2)) \). Therefore, since \(-2 < 0\), the graph of \( h \) should be the translation of the graph of \( f \) to the left by \(|-2| \) units.

- What concept from Grade 8 Geometry best describes the shifts of the graphs of the functions in Example 1?
  - Translation. In fact, we use the word “translate” to help you remember.

- Students should be comfortable explaining the difference between the translations of the graphs \( y = |x| + k \) and \( y = |x + k| \).

- Students may confuse the direction of a horizontal translation since the equation may seem to indicate the “opposite” direction (i.e., \( y = |x + 3| \) may be confused as a translation to the right because of the addition of 3 to \( x \)), especially since a vertical translation up is the transformation given by adding a positive number \( k \) to the function. Help students articulate why the horizontal translation behaves as it does.

- Consider the function \( g(x) = |x - 3| \) and its graph from Example 1. There is a point \((x + 3, g(x + 3))\) on the graph of \( g \). We have \( g(x + 3) = f(x + 3 - 3) = f(x) \). Then the point \((x + 3, f(x))\) is on the graph of \( g \). Since \((x, f(x))\) is on the graph of \( f \) and \((x + 3, f(x))\) is \((x, f(x))\) shifted 3 units to the right, we conclude that the graph of \( g \) is the graph of \( f \) translated 3 units to the right. A similar argument can be made for the graph of \( h \).

Exercises 1–3 (15 minutes)

Have students discuss the following four exercises in pairs. Discuss the answers as a class.

Exercise 1–3

1. Karla and Isamar are disagreeing over which way the graph of the function \( g(x) = |x + 3| \) is translated relative to the graph of \( f(x) = |x| \). Karla believes the graph of \( g \) is “to the right” of the graph of \( f \), Isamar believes the graph is “to the left.” Who is correct? Use the coordinates of the vertex of \( f \) and \( g \) to support your explanation.

   The graph of \( g \) is the graph of \( f \) translated to the left. The vertex of the graph of \( f \) is the point \((0, 0)\), whereas the vertex of the graph of \( g \) is the point \((-3, 0)\).

Note that in this lesson, students are working with translations of the function \( f(x) = |x| \). This function was chosen because it is one of the easier functions to use in showing how translations behave—just follow what happens to the vertex. We know that \((0, 0)\), or the vertex, is the point of the graph of \( f \) where the function’s outputs change between decreasing and increasing. As a horizontal translation, the vertex of the graph of \( g \) will also have a \( y \)-coordinate of 0; in fact, the vertex is \((-3, 0)\). Thus the graph of \( f \) is translated 3 units to the left to get the graph of \( g \).

2. Let \( f(x) = |x| \) where \( x \) can be any real number. Write a formula for the function whose graph is the transformation of the graph of \( f \) given by the instructions below.
   a. A translation right 5 units.
      \[ a(x) = |x - 5| \]
   b. A translation down 3 units.
      \[ b(x) = |x| - 3 \]
c. A vertical scaling (a vertical stretch) with scale factor of 5.
   \[ c(x) = 5|x| \]

   d. A translation left 4 units.
   \[ d(x) = |x + 4| \]

   e. A vertical scaling (a vertical shrink) with scale factor of \( \frac{1}{3} \).
   \[ e(x) = \frac{1}{3}|x| \]

3. Write the formula for the function depicted by the graph.

   a. \[ y = |x + 6| \]

   b. \[ y = -2|x| \]
c. \( y = |x - \frac{3}{2}| \)

d. \( y = |x| + 4 \)

e. \( y = \frac{1}{4}|x| \)
Exercises 4–5 (12 minutes)

Students now examine questions where more than one change is applied to $f(x) = |x|$.

### Exercises 4–5

4. Let $f(x) = |x|$ where $x$ can be any real number. Write a formula for the function whose graph is the described transformation of the graph of $f$.
   
   a. A translation 2 units left and 4 units down.
   
   $$y = |x + 2| - 4$$

   b. A translation 2.5 units right and 1 unit up.
   
   $$y = |x - 2.5| + 1$$

   c. A vertical scaling with scale factor $\frac{1}{2}$, and then a translation 3 units right.
   
   $$y = \frac{1}{2}|x - 3|$$

   d. A translation 5 units right and a vertical scaling by reflected across the $x$-axis with vertical scale factor $-2$.
   
   $$y = -2|x - 5|$$

5. Write the formula for the function depicted by the graph.

   a. $y = |x + 2| - 4$

   

   b. $y = |x - 5| - 2$
Closing (2 minutes)

- There is nothing special about using the function \( f(x) = |x| \) as we did in this lesson. The effects of these transformations on the graph of a function hold true for all functions.
- How can the graph of \( y = f(x) \) can be horizontally translated by positive or negative \( k \)?
- Draw a graph of a made-up function on the board, labeled by \( y = f(x) \), and show how to translate it right or left by \( k \) units using the equation \( y = f(x + k) \).

Exit Ticket (3 minutes)
Lesson 18: Four Interesting Transformations of Functions

Exit Ticket

Write the formula for the functions depicted by the graphs below:

a. \( f(x) = \) ________________

b. \( g(x) = \) ________________

c. \( h(x) = \) ________________
Exit Ticket Sample Solutions

Write the formula for the functions depicted by the graphs below:

- a. \( f(x) = |x - 5| - 4 \)
- b. \( g(x) = |x - 1| + 3 \)
- c. \( h(x) = |x + 6| - 2 \)

Problem Set Sample Solutions

1. Working with quadratic functions.
   a. The vertex of the quadratic function \( f(x) = x^2 \) is at \((0, 0)\), which is the minimum for the graph of \( f \). Based on your work in this lessons, to where do you predict the vertex will be translated for the graphs of \( g(x) = (x - 2)^2 \) and \( h(x) = (x + 3)^2 \)?

   *The vertex of \( g \) will be at \((2, 0)\); The vertex of \( h \) will be at \((-3, 0)\).*

   b. Complete the table of values and then graph all three functions.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = x^2 )</th>
<th>( g(x) = (x - 2)^2 )</th>
<th>( h(x) = (x + 3)^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>3</td>
<td>25</td>
<td>0</td>
</tr>
<tr>
<td>-2</td>
<td>2</td>
<td>16</td>
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</tr>
<tr>
<td>3</td>
<td>3</td>
<td>1</td>
<td>36</td>
</tr>
</tbody>
</table>

   ![Graphs of functions](image)
2. Let \( f(x) = |x - 4| \) for every real number \( x \). The graph of the equation \( y = f(x) \) is provided on the Cartesian plane below. Transformations of the graph of \( y = f(x) \) are described below. After each description, write the equation for the transformed graph. Then, sketch the graph of the equation you write for part (d).

   a. Translate the graph left 6 units and down 2 units.
      \[
      y = |x + 6| - 2 \quad \text{or} \quad y = f(x + 6) - 2
      \]

   b. Reflect the resulting graph from part (a) across the \( x \)-axis.
      \[
      y = -|x + 6| + 2 \quad \text{or} \quad y = -(f(x + 6) - 2)
      \]

   c. Scale the resulting graph from part (b) vertically by a scale factor of \( \frac{1}{2} \).
      \[
      y = -\frac{1}{2}|x + 6| + 1 \quad \text{or} \quad y = -\frac{1}{2}(f(x + 6) - 2)
      \]

   d. Translate the resulting graph from part (c) right 3 units and up 2 units. Graph the resulting equation.
      \[
      y = -\frac{1}{2}|x + 3| + 3 \quad \text{or} \quad y = -\frac{1}{2}(f(x + 3) - 2) + 2
      \]

3. Let \( f(x) = |x| \) for all real numbers \( x \). Write the formula for the function represented by the described transformation of the graph of \( y = f(x) \).

   a. First, a vertical stretch with scale factor \( \frac{1}{3} \) is performed, then a translation right 3 units, and finally a translation down 1 unit.
      \[
      a(x) = \frac{1}{3}|x - 3| - 1
      \]

   b. First, a vertical stretch with scale factor 3 is performed, then a reflection over the \( x \)-axis, then a translation left 4 units, and finally a translation up 5 units.
      \[
      b(x) = -3|x + 4| + 5
      \]
c. First, a reflection across the x-axis is performed, then a translation left 4 units, then a translation up 5 units, and finally a vertical stretch with scale factor 3.

\[ c(x) = -3|x + 4| + 15 \]

d. Compare your answers to parts (b) and (c). Why are they different?

In part (c), the vertical stretch happens at the end, which means the graph resulting from the first three transformations is what is vertically stretched: 

\[ c(x) = 3(-|x + 4| + 5) \]

4. Write the formula for the function depicted by each graph.

a. \[ a(x) = \frac{1}{2}|x - 1| - 3 \]

b. \[ b(x) = -2|x + 3| + 4 \]
Lesson 19: Four Interesting Transformations of Functions

Student Outcomes

- Students examine that a horizontal scaling with scale factor \( k \) of the graph of \( y = f(x) \) corresponds to changing the equation from \( y = f(x) \) to \( y = f\left( \frac{1}{k}x \right) \).

Lesson Notes

In this lesson, students study the effect a horizontal scaling by scale factor \( k \) has on the graph of an equation \( y = f(x) \). For example, if \( 0 < k < 1 \), a horizontal scaling by \( k \) will horizontally shrink any geometric figure in the Cartesian plane, including figures that are graphs of functions. The horizontal scaling of a graph corresponds to changing the equation from \( y = f(x) \) to \( y = f\left( \frac{1}{k}x \right) \). For values of scale factor \( k \) where \( k > 1 \), the graph of \( y = f\left( \frac{1}{k}x \right) \) is a horizontal stretch of the graph of \( y = f(x) \) by a factor of \( k \).

In this lesson, students may employ MP.3 when they make conjectures about the effect of \( k \), MP.8 when they use repeated reasoning to determine the effect of \( k \), and MP.6 when they communicate the effect to others using careful language.

Classwork

Students explore the horizontal scaling of the graph of \( y = f(x) \) when the equation changes from \( y = f(x) \) to \( y = f\left( \frac{1}{k}x \right) \) for \( 0 < k < 1 \). In this case, students see the graph of \( f \) is a horizontal “shrink” by \( k \). In Example 1, the scale factor for \( g \) is \( k = \frac{1}{2} \), or \( g(x) = f\left( \frac{1}{2}x \right) \), or \( g(x) = f(2x) \).

Example 1 (8 minutes)

Example 1

Let \( f(x) = x^2 \) and \( g(x) = f(2x) \), where \( x \) can be any real number.

a. Write the formula for \( g \) in terms of \( x^2 \) (i.e., without using \( f(x) \) notation):

\[ g(x) = (2x)^2 \]
b. Complete the table of values for these functions.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x) = x^2$</th>
<th>$g(x) = f(2x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>9</td>
<td>36</td>
</tr>
<tr>
<td>-2</td>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>36</td>
</tr>
</tbody>
</table>

c. Graph both equations: $y = f(x)$ and $y = f(2x)$.

See the discussion below for an explanation of the steps and arrows.

d. How does the graph of $y = g(x)$ relate to the graph of $y = f(x)$?

The corresponding $x$-value of $y = g(x)$ is half of the corresponding $x$-value of $y = f(x)$ when $g(x) = f(x)$, the points of the graph of $g$ are $\frac{1}{2}$ the distance to the $y$-axis as the corresponding points of the graph of $f$, which makes the graph of $g$ appear to “shrink horizontally.”

e. How are the values of $f$ related to the values of $g$?

For equal outputs of $f$ and $g$, the input of $g$ only has to be half as big as the input of $f$. 
Discussion (5 minutes)

- A horizontal scaling of a graph with scale factor $\frac{1}{2}$ will “shrink” the original graph $y = f(x)$ horizontally by $\frac{1}{2}$ and correspond to the graph of the equation $y = f\left(\frac{1}{2}x\right)$ or $y = f(2x)$, i.e., the horizontal scaling of the graph of $y = f(x)$ with scale factor $k > 0$ is the graph of the equation $y = f\left(\frac{1}{k}x\right)$.

- In Example 1, what process could be used to find the value of $g(x)$ for any given number $x$, using only the graph of $y = f(x)$ (not the formula for $f(x)$)?
  - **Step 1:** Find $x$ on the $x$-axis.
  - **Step 2:** Multiply $x$ by 2 to find the number $2x$ on the $x$-axis.
  - **Step 3:** Find the value of $f$ at $2x$.
  - **Step 4:** Move parallel to the $x$-axis from the point found in Step 3 until directly over/under/on $x$. That point is $(x, g(x))$. [These steps are numbered and illustrated in the graph above for $x = 1$.]

Lightly erase the graph of $y = g(x)$ (already drawn from part (c)), and then go through the steps above to redraw it, picking out a few points to help students see that only the $y$-values are changing between corresponding points on the graph of $f$ and the graph of $g$. If you erased the graph lightly enough so that the “ghost” of the image is still there, students will see that you are redrawing the graph of $g$ over the original graph. Following the steps will give students a sense of how the points of the graph of $f$ are only “shrinking” in the $x$-values, not the $y$-values.

Many students might confuse a horizontal scaling with other types of transformations like dilations. In fact, a dilation with scale factor $\frac{1}{4}$ of the graph of $f$ in this example produces the exact same image as a horizontal scaling by $\frac{1}{2}$, but the correspondence between the points is different. Your goal in Grade 9 is to have students develop a “rigid” notion of what a vertical scaling means so that it can be profitably compared to dilation in Grades 10 and 11.

- Consider a function $f$, and a transformation of that function $h$, such that $h(x) = f\left(\frac{1}{k}x\right)$. how do the domain and range of $f$ relate to the domain and range of $h$?
  - **The range of both functions will be the same, but the domains may change.**

- What might the graph of $y = f(1,000x)$ look like?

- What might the graph of $y = f(1,000,000x)$ look like if it were graphed on the same Cartesian plane as the graphs of $f$ and $g$?

Let students go up to the board and draw their conjectures on the plane.

Discussion (5 minutes)

Students explore the horizontal scaling of the graph of $y = f(x)$ when the equation changes from $y = f(x)$ to $y = f\left(\frac{1}{k}x\right)$ for $k > 1$. In this case, students see that the graph of $f$ is horizontally “stretched” by a factor of $k$. In Example 2, the scale factor for $g$ is $k = 2$, or $g(x) = f\left(\frac{1}{2}x\right)$. 
Example 2 (8 minutes)

Example 2

Let \( f(x) = x^2 \) and \( h(x) = f\left(\frac{1}{2}x\right) \), where \( x \) can be any real number.

a. Rewrite the formula for \( h \) in terms of \( x^2 \) (i.e., without using \( f(x) \) notation):

\[
h(x) = \left(\frac{1}{2}x\right)^2
\]

b. Complete the table of values for these functions.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = x^2 )</th>
<th>( h(x) = f\left(\frac{1}{2}x\right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>9</td>
<td>2.25</td>
</tr>
<tr>
<td>-2</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>0.25</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.25</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>2.25</td>
</tr>
</tbody>
</table>

c. Graph both equations: \( y = f(x) \) and \( y = f\left(\frac{1}{2}x\right) \).
Lesson 19: Four Interesting Transformations of Functions

How does the graph of \( y = f(x) \) relate to the graph of \( y = h(x) \)?

Since the corresponding \( x \)-value of \( y = h(x) \) is twice the corresponding \( x \)-value of \( y = f(x) \) when \( g(x) = f(x) \), the points of the graph of \( g \) are 2 times the distance to the \( y \)-axis as the corresponding points of the graph of \( f \), which makes the graph of \( g \) appear to “stretch horizontally.”

How are the values of \( f \) related to the values of \( h \)?

To get equal outputs of each function, the input of \( h \) has to be twice the input of \( f \).

A horizontal scale of a graph with scale factor 2 will "stretch" the original graph \( y = f(x) \) horizontally by 2 and correspond to the graph of the equation \( y = f\left(\frac{1}{2}x\right) \), i.e., the horizontal scale of the graph of \( y = f(x) \) with scale factor \( k > 0 \) is once again the graph of the equation \( y = f\left(\frac{1}{k}x\right) \). Follow the steps given in Discussion 1 to show students how to find the value \( h(x) \) on the Cartesian plane using only the graph of \( f \) (not the formula for \( f \)). Emphasize that only the \( y \)-values are being scaled. When comparing \( y = f(x) \) to \( y = f\left(\frac{1}{k}x\right) \), the range of both functions will be the same, but the domains may change. Ask students what the graph of \( f \) might look like after a horizontal scale with scale factor \( k = 10000 \). Let them draw their conjecture on the graph on the board. Then ask them what the equation of the resulting graph is.

Exercise 1 (6 minutes)

Have students discuss the following exercise in pairs. Discuss the answer as a class.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = 2^x )</th>
<th>( g(x) = 2^{(2x)} )</th>
<th>( h(x) = 2^{(-x)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( -2 )</td>
<td>( \frac{1}{4} )</td>
<td>( \frac{1}{16} )</td>
<td>4</td>
</tr>
<tr>
<td>( -1 )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{4} )</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>4</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>16</td>
<td>( \frac{1}{4} )</td>
</tr>
</tbody>
</table>
Lesson 19

Four Interesting Transformations of Functions

b. Label each of the graphs with the appropriate functions from the table.

\[
\begin{align*}
y &= f(x) \\
y &= g(x) \\
y &= h(x)
\end{align*}
\]

c. Describe the transformation that takes the graph of \( y = f(x) \) to the graph of \( y = g(x) \).

\[\text{The graph of } y = g(x) \text{ is a horizontal scale with scale factor } \frac{1}{2} \text{ of the graph of } y = f(x).\]

d. Consider \( y = f(x) \) and \( y = h(x) \). What does negating the input do to the graph of \( f \)?

\[\text{The graph of } h \text{ is a reflection over the } y\text{-axis of the graph of } f.\]

e. Write the formula of an exponential function whose graph would be a horizontal stretch relative to the graph of \( g \).

Answers will vary. Example: \( y = 2^{0.5x} \).

Example 3 (6 minutes)

Example 3

a. Look at the graph of \( y = f(x) \) for the function \( f(x) = x^2 \) in Example 1 again. Would we see a difference in the graph of \( y = g(x) \) if \(-2\) was used as the scale factor instead of \(2\)? If so, describe the difference. If not, explain why not.

There would be no difference. The function involves squaring the value within the parentheses, so the graph of \( y = f(2x) \) and the graph of \( y = f(-2x) \) both will be the same as the graph of \( y = g(x) \), but both correspond to different transformations: The first is a horizontal scaling with scale factor \( \frac{1}{2} \) and the second is a horizontal scaling with scale factor \( \frac{1}{2} \) and a reflection across the \( y\)-axis.
b. A reflection across the $y$-axis takes the graph of $y = f(x)$ for the function $f(x) = x^2$ back to itself. Such a transformation is called a reflection symmetry. What is the equation for the graph of the reflection symmetry of the graph of $y = f(x)$?

$y = f(-x)$.

Tell students that if a function satisfies the equation $f(x) = f(-x)$ for every number $x$ in the domain of $f$, it is called an even function. A consequence of an even function is that its graph is symmetrical with respect to the $y$-axis. Furthermore, the graph of $f(x) = x^2$ is symmetrical across the $y$-axis. A reflection across the $y$-axis does not change the graph.

c. Deriving the answer to the following question is fairly sophisticated; do only if you have time: In Lessons 17 and 18, we used the function $f(x) = |x|$ to examine the graphical effects of transformations of a function. Here in Lesson 19, we use the function $f(x) = x^2$ to examine the graphical effects of transformations of a function. Based on the observations you made while graphing, why would using $f(x) = x^2$ be a better option than using the function $f(x) = |x|$?

Not all of the effects of multiplying the input of a function are as visible with an absolute function as it is with a quadratic function. For example, the graph of $y = 2|x|$ is the same as $y = |2x|$. Therefore, it is easier to see the effect of multiplying a value to the input of a function by using a quadratic function than it is by using the absolute value function.

Closing (2 minutes)

Discuss how the horizontal scaling by a scale factor of $k$ of the graph of a function $y = f(x)$ corresponds to changing the equation of the graph from $y = f(x)$ to $y = f\left(\frac{1}{k}x\right)$. Investigate the four cases of $k$:

1. $k > 1$
2. $0 < k < 1$
3. $-1 < k < 0$
4. $k < -1$

Exit Ticket (5 minutes)
Lesson 19: Four Interesting Transformations of Functions

Exit Ticket

Let \( f(x) = x^2 \), \( g(x) = (3x)^2 \), and \( h(x) = \left(\frac{1}{3}x\right)^2 \), where \( x \) can be any real number. The graphs above are of \( y = f(x) \), \( y = g(x) \), and \( y = h(x) \).

1. Label each graph with the appropriate equation.

2. Describe the transformation that takes the graph of \( y = f(x) \) to the graph of \( y = g(x) \). Use coordinates of each to illustrate an example of the correspondence.
3. Describe the transformation that takes the graph of \( y = f(x) \) to the graph of \( y = h(x) \). Use coordinates to illustrate an example of the correspondence.
Exit Ticket Sample Solutions

Let \( f(x) = x^2 \), \( g(x) = (3x)^2 \), and \( h(x) = \left(\frac{1}{3} x\right)^2 \), where \( x \) can be any real number. The graphs above are of \( y = f(x) \), \( y = g(x) \), and \( y = h(x) \).

1. Label each graph with the appropriate equation.
   
   See graph.

2. Describe the transformation that takes the graph of \( y = f(x) \) to the graph of \( y = g(x) \). Use coordinates of each to illustrate an example of the correspondence.

   The graph of \( y = g(x) \) is a horizontal shrink of the graph of \( y = f(x) \) with scale factor \( \frac{1}{3} \). The corresponding \( x \)-value of \( y = g(x) \) is one-third of the corresponding \( x \)-value of \( y = f(x) \) when \( g(x) = f(x) \). This can be illustrated with the coordinate \((1, 9)\) on \( g(x) \) and the coordinate \((3, 9)\) on \( f(x) \).

3. Describe the transformation that takes the graph of \( y = f(x) \) to the graph of \( y = h(x) \). Use coordinates to illustrate an example of the correspondence.

   The graph of \( h(x) \) is a horizontal stretch of the graph of \( f(x) \) with scale factor \( 3 \). The corresponding \( x \)-value of \( y = h(x) \) is three times the corresponding \( x \)-value of \( y = f(x) \) when \( h(x) = f(x) \). This can be illustrated with the coordinate \((1, 1)\) on \( f(x) \) and the coordinate \((3, 1)\) on \( h(x) \).
Problem Set Sample Solutions

Let \( f(x) = x^2 \), \( g(x) = 2x^2 \), and \( h(x) = (2x)^2 \), where \( x \) can be any real number. The graphs above are of the functions \( y = f(x) \), \( y = g(x) \), and \( y = h(x) \).

1. Label each graph with the appropriate equation.
   
   \[
   y = f(x) \quad y = g(x) \quad y = h(x)
   \]

2. Describe the transformation that takes the graph of \( y = f(x) \) to the graph of \( y = g(x) \). Use coordinates to illustrate an example of the correspondence.

   \[y = g(x) \text{ is a vertical stretch of } y = f(x) \text{ by scale factor } 2; \text{ for a given } x\text{-value, the value of } g(x) \text{ is twice as much as the value of } f(x).\]
   
   OR:

   \[\text{The graph of } y = g(x) \text{ is a horizontal shrink of } y = f(x) \text{ by scale factor } \frac{1}{\sqrt{2}}; \text{ it takes } \frac{1}{\sqrt{2}} \text{ times the input for } y = g(x) \text{ as compared to } y = f(x) \text{ to yield the same output.}\]

3. Describe the transformation that takes the graph of \( y = f(x) \) to the graph of \( y = h(x) \). Use coordinates to illustrate an example of the correspondence.

   \[y = h(x) \text{ is a horizontal shrink of } y = f(x) \text{ by scale factor } \frac{1}{2}; \text{ it takes } \frac{1}{2} \text{ the input for } y = h(x) \text{ as compared to } y = f(x) \text{ to yield the same output.}\]
   
   OR:

   \[\text{The graph of } y = h(x) \text{ is a vertical stretch of } y = f(x) \text{ by scale factor } 4; \text{ for a given } x\text{-value, the value of } h(x) \text{ is four times as much as the value of } f(x).\]
Lesson 20: Four Interesting Transformations of Functions

Student Outcomes

- Students apply their understanding of transformations of functions and their graphs to piecewise functions.

Lesson Notes

In Lessons 17–19 students study translations and scalings of functions and their graphs. In Lesson 20, these transformations are applied in combination to piecewise functions. Students should become comfortable visualizing how the graph of a transformed piecewise function will relate to the graph of the original piecewise function.

Classwork

Opening Exercise (6 minutes)

Have students work individually or in pairs to complete the opening exercise. This exercise highlights MP.7 since it calls on students to interpret the meaning of $k$ in the context of a graph.

<table>
<thead>
<tr>
<th>Graph of $y = f(x)$</th>
<th>Vertical</th>
<th>Horizontal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Translate $y = f(x) + k$</td>
<td>$k &gt; 0$ Translate up by $</td>
<td>k</td>
</tr>
<tr>
<td>$k &lt; 0$ Translate down by $</td>
<td>k</td>
<td>$ units</td>
</tr>
<tr>
<td>Scale by scale factor $k$</td>
<td>$k &gt; 1$ Vertical stretch by a factor of $</td>
<td>k</td>
</tr>
<tr>
<td>$0 &lt; k &lt; 1$ Vertical shrink by a factor of $</td>
<td>k</td>
<td>$</td>
</tr>
<tr>
<td>$-1 &lt; k &lt; 0$ Vertical shrink by a factor of $</td>
<td>k</td>
<td>$ and reflection over x-axis</td>
</tr>
<tr>
<td>$k &lt; -1$ Vertical stretch by a factor of $</td>
<td>k</td>
<td>$ and reflection over x-axis</td>
</tr>
</tbody>
</table>
In Lesson 15, we discovered how the absolute value function can be written as a piecewise function. Example 1 and the associated exercises are intended to help students reexamine how piecewise functions behave.

Example 1 (3 minutes)

A transformation of the absolute value function, \( f(x) = |x - 3| \), is rewritten here as a piecewise function. Describe in words how to graph this piecewise function.

\[
f(x) = \begin{cases} 
-x + 3, & x < 3 \\
3, & x \geq 3 
\end{cases}
\]

First, I would graph the line \( y = -x + 3 \) for \( x \)-values less than 3 and then I would graph the line \( y = x - 3 \) for \( x \)-values greater than or equal to 3.

Exercises 1–2 (15 minutes)

1. Describe how to graph the following piecewise function. Then graph \( y = f(x) \) below.

\[
f(x) = \begin{cases} 
-3x - 3, & x \leq -2 \\
0.5x + 4, & -2 < x < 2 \\
-2x + 9, & x \geq 2 
\end{cases}
\]

The function \( f \) can be graphed of as the line \( y = -3x - 3 \) for \( x \)-values less than or equal to \(-2\), the graph of the line \( y = 0.5x + 4 \) for \( x \)-values greater than \(-2\) and less than \(2\), and the graph of the line \( y = -2x + 9 \) for \( x \)-values greater than or equal to \(2\).
2. Using the graph of $f$ below, write a formula for $f$ as a piecewise function.

$$f(x) = \begin{cases} 
0.5x - 0.5, & -7 \leq x \leq -3 \\
-2, & -3 < x < 1 \\
x - 3, & 1 \leq x \leq 4 \\
5 - x, & 4 < x \leq 7 
\end{cases}$$

Or

$$f(x) = \begin{cases} 
0.5x - 0.5, & -7 \leq x \leq -3 \\
-2, & -3 < x < 1 \\
-|x - 4| + 1, & 1 \leq x \leq 7
\end{cases}$$

Example 2 (10 minutes)

Students translate and scale the graph of a piecewise function.

Example 2

The graph $y = f(x)$ of a piecewise function $f$ is shown. The domain of $f$ is $-5 \leq x \leq 5$, and the range is $-1 \leq y \leq 3$.

a. Mark and identify four strategic points helpful in sketching the graph of $y = f(x)$.

$(-5, -1), (-1, 1), (3, 1), and (5, 3)$
b. Sketch the graph of $y = 2f(x)$ and state the domain and range of the transformed function. How can you use part (a) to help sketch the graph of $y = 2f(x)$?

Domain: $-5 \leq x \leq 5$, range: $-2 \leq y \leq 6$. For every point $(x, y)$ in the graph of $f(x)$, there is a point $(x, 2y)$ on the graph of $y = 2f(x)$. The four strategic points can be used to determine the line segments in the graph of $y = 2f(x)$ by graphing points with the same original $x$-coordinate and 2 times the original $y$-coordinate ($(-5, -2), (-1, 2), (3, 2)$, and $(5, 6)$).

c. A horizontal scaling with scale factor $\frac{1}{2}$ of the graph of $y = f(x)$ is the graph of $y = f(2x)$. Sketch the graph of $y = f(2x)$ and state the domain and range. How can you use the points identified in part (a) to help sketch $y = f(2x)$?

Domain: $-2.5 \leq x \leq 2.5$, range: $-1 \leq y \leq 3$. For every point $(x, y)$ in the graph of $f(x)$, there is a point ($\frac{x}{2}, y$) on the graph of $y = f(2x)$. The four strategic points can be used to determine the line segments in the graph of $y = f(2x)$ by graphing points with one-half the original $x$-coordinate and the original $y$-coordinate ($(-2.5, -1), (-0.5, 1), (1.5, 1)$, and $(2.5, 3)$).
Exercises 3–4 (5 minutes)

3. How does the range of \( f \) in Example 2 compare to the range of a transformed function \( g \), where \( g(x) = kf(x) \), when \( k > 1 \)?

   For every point \((x, y)\) in the graph of \( y = f(x) \), there is a point \((x, ky)\) in the graph of \( y = kf(x) \), where the number \( ky \) is a multiple of each \( y \). For values of \( k > 1 \), \( y = kf(x) \) is a vertical scaling that appears to stretch the graph of \( y = f(x) \). The original range, \(-1 \leq y \leq 3\) for \( y = f(x) \) becomes \(-1k \leq y \leq 3k\) for the function \( y = kf(x) \).

4. How does the domain of \( f \) in Example 2 compare to the domain of a transformed function \( g \), where \( g(x) = f\left(\frac{1}{k}x\right) \), when \( 0 < k < 1 \)? (Hint: How does a graph shrink when it is horizontally scaled by a factor \( k \)?)

   For every point \((x, y)\) in the graph of \( y = f(x) \), there is a point \((kx, y)\) in the graph of \( y = f\left(\frac{1}{k}x\right) \). For values of \( 0 < k < 1 \), \( y = f\left(\frac{1}{k}x\right) \) is a horizontal scaling by a factor \( k \) that appears to shrink the graph of \( y = f(x) \). This means the original domain, \(-5 \leq x \leq 5\) for \( y = f(x) \) becomes \(-5k \leq x \leq 5k\) for the function \( y = f\left(\frac{1}{k}x\right) \).

Closing (2 minutes)

- The transformations that translate and scale familiar functions, like the absolute value function, also apply to piecewise functions and to any function in general.
- By focusing on strategic points in the graph of a piecewise function, we can translate and scale the entire graph of the function by manipulating the coordinates of those few points.

Exit Ticket (4 minutes)
Lesson 20: Four Interesting Transformations of Functions

Exit Ticket

The graph of a piecewise function $f$ is shown below.

Let $p(x) = f(x - 2)$, $q(x) = \frac{1}{2}f(x - 2)$, and $r(x) = \frac{1}{2}f(x - 2) + 3$.

Graph $y = p(x)$, $y = q(x)$, and $y = r(x)$ on the same set of axes as the graph of $y = f(x)$.
Exit Ticket Sample Solutions

The graph of a piecewise function $f$ is shown below.

Let $p(x) = f(x - 2)$, $q(x) = \frac{1}{2} f(x - 2)$, and $r(x) = \frac{1}{2} f(x - 2) + 3$.

Graph $y = p(x)$, $y = q(x)$, and $y = r(x)$ on the same set of axes as the graph of $y = f(x)$.
Problem Set Sample Solutions

1. Suppose the graph of \( f \) is given. Write an equation for each of the following graphs after the graph of \( f \) has been transformed as described.
   
   a. Translate 5 units upward.
      \[ y = f(x) + 5 \]

   b. Translate 3 units downward.
      \[ y = f(x) - 3 \]

   c. Translate 2 units right.
      \[ y = f(x - 2) \]

   d. Translate 4 units left.
      \[ y = f(x + 4) \]

   e. Reflect about the \( x \)-axis.
      \[ y = -f(x) \]

   f. Reflect about the \( y \)-axis.
      \[ y = f(-x) \]

   g. Stretch vertically by a factor of 2.
      \[ y = 2f(x) \]

   h. Shrink vertically by a factor of \( \frac{1}{3} \).
      \[ y = \frac{1}{3}f(x) \]

   i. Shrink horizontally by a factor of \( \frac{1}{3} \).
      \[ y = f(3x) \]

   j. Stretch horizontally by a factor of 2.
      \[ y = f\left(\frac{1}{2}x\right) \]

2. Explain how the graphs of the equations below are related to the graph of \( y = f(x) \).
   
   a. \( y = 5f(x) \)
      
      The graph is a vertical stretch of \( y = f(x) \) by a factor of 5.
b. \[ y = f(x - 4) \]

The graph of \( y = f(x) \) is translated right 4 units.

c. \[ y = -2f(x) \]

The graph is a vertical stretch of \( y = f(x) \) by a factor of 2 and reflected about the x-axis.

d. \[ y = f(3x) \]

The graph is a horizontal shrink of \( y = f(x) \) by a factor of \( \frac{1}{3} \).

e. \[ y = 2f(x) - 5 \]

The graph is a vertical stretch of \( y = f(x) \) by a factor of 2 and translated down 5 units.

3. The graph of the equation \( y = f(x) \) is provided below. For each of the following transformations of the graph, write a formula (in terms of \( f \)) for the function that is represented by the transformation of the graph of \( y = f(x) \). Then draw the transformed graph of the function on the same set of axes as the graph of \( y = f(x) \).

   a. A translation 3 units left and 2 units up.
      \[ p(x) = f(x) + 2 \]
Lesson 20: Four Interesting Transformations of Functions

b. A vertical stretch by a scale factor of 3.
   \[ q(x) = 3f(x) \]

c. A horizontal shrink by a scale factor of \( \frac{1}{2} \).
   \[ r(x) = f(2x) \]

4. Reexamine your work on Example 2 and Exercises 3 and 4 from this lesson. The questions in (b) and (c) of Example 2 asked how the equations \( y = 2f(x) \) and \( y = f(2x) \) could be graphed with the help of the strategic points found in (a). In this problem, we investigate whether it is possible to determine the graphs of \( y = 2f(x) \) and \( y = f(2x) \) by working with the piecewise-linear function \( f \) directly.

a. Write the function \( f \) in Example 2 as a piecewise-linear function.
   \[
   f(x) = \begin{cases} 
   0.5x + 1.5, & -5 \leq x \leq -1 \\
   1, & -1 < x < 3 \\
   x - 2, & 3 \leq x \leq 5 
   \end{cases}
   \]
b. Let $g(x) = 2f(x)$. Use the graph you sketched in Example 2(b) of $y = 2f(x)$ to write the formula for the function $g$ as a piecewise-linear function.

\[
g(x) = \begin{cases} 
 x + 3, & -5 \leq x < -1 \\
 2, & -1 < x < 3 \\
 2x - 4, & 3 \leq x \leq 5 
\end{cases}
\]

c. Let $h(x) = f(2x)$. Use the graph you sketched in Example 2© of $y = f(2x)$ to write the formula for the function $h$ as a piecewise-linear function.

\[
h(x) = \begin{cases} 
 x + 1.5, & -2.5 \leq x \leq -0.5 \\
 1, & -0.5 < x < 1.5 \\
 2x - 2, & 1.5 \leq x \leq 2.5 
\end{cases}
\]

d. Compare the piecewise linear functions $g$ and $h$ to the piecewise linear function $f$. Did the expressions defining each piece change? If so, how? Did the domains of each piece change? If so how?

Function $g$: Each piece of the formula for $g$ is 2 times the corresponding piece of the formula for $f$. The domains are the same.

Function $h$: Each piece of the formula for $h$ is found by substituting $2x$ in for $x$ in the corresponding piece of the formula for $f$. The length of each interval in the domain of $h$ is $\frac{1}{2}$ the length of the corresponding interval in the domain of $f$. 

1. Given \( h(x) = |x + 2| - 3 \) and \( g(x) = -|x| + 4 \).
   
a. Describe how to obtain the graph of \( g \) from the graph of \( a(x) = |x| \) using transformations.

   b. Describe how to obtain the graph of \( h \) from the graph of \( a(x) = |x| \) using transformations.

   c. Sketch the graphs of \( h(x) \) and \( g(x) \) on the same coordinate plane.

   d. Use your graphs to estimate the solutions to the equation:
      \[ |x + 2| - 3 = -|x| + 4 \]
      Explain how you got your answer.

   e. Were your estimations you made in part (d) correct? If yes, explain how you know. If not explain why not.
2. Let $f$ and $g$ be the functions given by $f(x) = x^2$ and $g(x) = x|x|$. 

a. Find $f\left(\frac{1}{3}\right)$, $g(4)$, and $g(-\sqrt{3})$. 

b. What is the domain of $f$? 

c. What is the range of $g$? 

d. Evaluate $f(-67) + g(-67)$. 

e. Compare and contrast $f$ and $g$. How are they alike? How are they different? 

f. Is there a value of $x$, such that $f(x) + g(x) = -100$? If so, find $x$. If not, explain why no such value exists. 

g. Is there a value of $x$ such that $(x) + g(x) = 50$? If so, find $x$. If not, explain why no such value exists.
3. A boy bought 6 guppies at the beginning of the month. One month later the number of guppies in his tank had doubled. His guppy population continued to grow in this same manner. His sister bought some tetras at the same time. The table below shows the number of tetras, \( t \), after \( n \) months have passed since they bought the fish.

<table>
<thead>
<tr>
<th>( n ), months</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t ), tetras</td>
<td>8</td>
<td>16</td>
<td>24</td>
<td>32</td>
</tr>
</tbody>
</table>

a. Create a function \( g \) to model the growth of the boy's guppy population, where \( g(n) \) is the number of guppies at the beginning of each month, and \( n \) is the number of months that have passed since he bought the 6 guppies. What is a reasonable domain for \( g \) in this situation?

b. How many guppies will there be one year after he bought the 6 guppies?

c. Create an equation that could be solved to determine how many months after he bought the guppies there will be 100 guppies.

d. Use graphs or tables to approximate a solution to the equation from part (c). Explain how you arrived at your estimate.
e. Create a function, \( t \), to model the growth of the sister’s tetra population, where \( t(n) \) is the number of tetras after \( n \) months have passed since she bought the tetras.

f. Compare the growth of the sister’s tetra population to the growth of the guppy population. Include a comparison of the average rate of change for the functions that model each population’s growth over time.

g. Use graphs to estimate the number of months that will have passed when the population of guppies and tetras will be the same.
h. Use graphs or tables to explain why the guppy population will eventually exceed the tetra population even though there were more tetras to start with.

i. Write the function \( g(n) \) in such a way that the percent increase in the number of fish per month can be identified. Circle or underline the expression representing percent increase in number of fish per month.
4. Regard the solid dark equilateral triangle as figure 0. Then, the first figure in this sequence is the one composed of three dark triangles, the second figure is the one composed of nine dark triangles, and so on.

![Figure 0](image1.png)  ![Figure 1](image2.png)  ![Figure 2](image3.png)  ![Figure 3](image4.png)  ![Figure 4](image5.png)

a. How many dark triangles are in each figure? Make a table to show this data.

<table>
<thead>
<tr>
<th>$n$ (Figure Number)</th>
<th></th>
<th></th>
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</thead>
<tbody>
<tr>
<td>$T$ (# of dark triangles)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b. Describe in words how, given the number of dark triangles in a figure, to determine the number of dark triangles in the next figure.

c. Create a function that models this sequence. What is the domain of this function?

d. Suppose the area of the solid dark triangle in Figure 0 is 1 square meter. The areas of one dark triangle in each figure form a sequence. Create an explicit formula that gives the area of just one of the dark triangles in the $n^{th}$ figure in the sequence?
e. The sum of the areas of all the dark triangles in Figure 0 is $1 \text{ m}^2$; there is only one triangle in this case. The sum of the areas of all the dark triangles in Figure 1 is $\frac{3}{4} \text{ m}^2$. What is the sum of the areas of all the dark triangles in the $n^{th}$ figure in the sequence? Is this total area increasing or decreasing as $n$ increases?

f. Let $P(n)$ be the sum of the perimeters of all the dark triangles in the $n^{th}$ figure in the sequence of figures. There is a real number $k$ so that:

$$P(n + 1) = kP(n)$$

is true for each positive whole number $n$. What is the value of $k$?
5. The graph of a piecewise function \( f \) is shown to the right. The domain of \( f \) is \(-3 \leq x \leq 3\).

   a. Create an algebraic representation for \( f \). Assume that the graph of \( f \) is composed of straight line segments.

   b. Sketch the graph of \( y = 2f(x) \) and state the domain and range.
c. Sketch the graph of \( y = f(2x) \) and state the domain and range.

d. How does the range of \( y = f(x) \) compare to the range of \( y = kf(x) \), where \( k > 1 \)?

e. How does the domain of \( y = f(x) \) compare to the domain of \( y = f(kx) \), where \( k > 1 \)?
<table>
<thead>
<tr>
<th>Assessment Task Item</th>
<th>STEP 1</th>
<th>STEP 2</th>
<th>STEP 3</th>
<th>STEP 4</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1</strong></td>
<td>Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem</td>
<td>Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem</td>
<td>A correct answer with some evidence of reasoning or application of mathematics to solve the problem, or an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem</td>
<td>A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem</td>
</tr>
<tr>
<td>a F-BF.B.3</td>
<td>Student answer is missing or entirely incorrect.</td>
<td>Student’s descriptions of transformations are partially correct.</td>
<td>Student’s descriptions of the transformations are correct, but there may be some minor misuse or omission of appropriate vocabulary.</td>
<td>Student’s descriptions of the transformations are clear and correct and use appropriate vocabulary.</td>
</tr>
<tr>
<td>b F-BF.B.3</td>
<td>Student answer is missing or entirely incorrect.</td>
<td>Student’s descriptions of transformations are partially correct.</td>
<td>Student’s descriptions of the transformations are correct, but there may be some minor misuse or omission of appropriate vocabulary.</td>
<td>Student’s descriptions of the transformations are clear and correct and use appropriate vocabulary.</td>
</tr>
<tr>
<td>c-e A-REI.D.11 F-BF.B.3</td>
<td>Student’s sketches do not resemble absolute value functions, and/or student is unable to use the graphs to estimate the solutions to the equation. Student may or may not have arrived at correct solutions of the equation via another method such as trial and error.</td>
<td>Student’s sketches resemble the graph of an absolute value function, but are inaccurate. Student shows evidence of using the intersection point of the graphs to find the solution but is unable to confirm his or her solution points; therefore, the conclusion in (e) is inconsistent with the intersection points.</td>
<td>Student sketches are accurate with no more than one minor error; the student shows evidence of using the intersection points to find the solutions to the equation. The conclusion in (e) is consistent with their estimated solutions but may have one error. Student’s communication is clear but could include more appropriate use of vocabulary or more detail.</td>
<td>Student sketches are accurate and solutions in part (d) match the x-coordinates of the intersection points. The student’s explanation for part (d) reflects an understanding that the process is analogous to solving the system $y = h(x)$ and $y = g(x)$. The work shown in part (e) supports his or her conclusion that estimates were or were not solutions and includes supporting explanation using appropriate vocabulary.</td>
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<tr>
<td>2</td>
<td>a</td>
<td>F-IF.A.2</td>
<td>Student provides no correct answers.</td>
<td>Student provides only one correct answer.</td>
</tr>
<tr>
<td></td>
<td>b - c</td>
<td>F-IF.A.1</td>
<td>Neither domain nor range is correct.</td>
<td>One of the two is identified correctly, or the student has reversed the ideas, giving the range of $f$ when asked for domain of $f$, and the domain of $g$ when asked for the range of $g$.</td>
</tr>
<tr>
<td>d</td>
<td>F-IF.A.2</td>
<td>Student makes a major error or omission in evaluating the expression (e.g., doesn’t substitute -67 into $f$ or $g$)</td>
<td>Student makes one or more errors in evaluating the expression.</td>
<td>Student evaluates the expression correctly, but work to support the answer is limited, or there is one minor error present.</td>
</tr>
<tr>
<td>e</td>
<td>F-IF.A.1 F-IF.A.2 F-IF.C.7a</td>
<td>Student makes little or no attempt to compare the two functions.</td>
<td>Student’s comparison does not note the similarity of the two functions yielding identical outputs for positive inputs and opposite outputs for negative inputs; it may be limited to superficial features, such as one involves squaring and the other contains an absolute value.</td>
<td>Student recognizes that they are equal for $x = 0$ and positive $x$-values but may not clearly articulate that the two functions are opposites when $x$ is negative.</td>
</tr>
<tr>
<td>f</td>
<td>F-IF.A.1 F-IF.A.2</td>
<td>Student provides an incorrect conclusion, OR makes little or no attempt to answer.</td>
<td>Student identifies that there is no solution but provides little or no supporting work or explanation.</td>
<td>Student identifies that there is no solution and provides an explanation, but the explanation is limited or contains minor inconsistencies or errors.</td>
</tr>
<tr>
<td>g</td>
<td>F-IF.A.1 F-IF.A.2</td>
<td>Student provides an incorrect conclusion, OR makes little or no attempt to answer.</td>
<td>Student identifies that $x = 5$ is a solution but provides little or no supporting work or explanation.</td>
<td>Student identifies that $x = 5$ is a solution and provides an explanation, but the explanation is limited or contains minor inconsistencies/ errors.</td>
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<tr>
<td>3</td>
<td>a</td>
<td>A-CED.A.1 F-BF.A.1a F-IF.B.5</td>
<td>Student does not provide an exponential function, OR student provides an exponential function that does not model the data the</td>
<td>Student provides a correct exponential function, but the domain is incorrect or omitted, OR student provides an exponential function that models the data and a domain that fits the situation.</td>
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<tr>
<td><strong>domain is incorrect or omitted.</strong></td>
<td>that does not model the data but correctly identifies the domain in this situation.</td>
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<tr>
<td>b</td>
<td><strong>F-IF.A.2</strong></td>
<td>Student gives an incorrect answer with no supporting calculations.</td>
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<td>Student gives an incorrect answer, but the answer is supported with the student’s function from part (a).</td>
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<td>Student has a minor calculation error in arriving at the answer. Student provides supporting work.</td>
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<td>Student provides a correct answer with proper supporting work.</td>
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<td>c</td>
<td><strong>F-BF.A.1a</strong></td>
<td>Student provides no equation or gives an equation that does not demonstrate understanding of what is required to solve the problem described.</td>
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<td>Student sets up an incorrect equation that demonstrates limited understanding of what is required to solve the problem described.</td>
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<td>Student provides a correct answer but then simplifies it into an incorrect equation, OR student has a minor error in the equation given but demonstrates substantial understanding of what is required to solve the problem.</td>
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<td></td>
<td>Student provides a correct equation that demonstrates understanding of what is required to solve the problem.</td>
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<td>d</td>
<td><strong>F-IF.A.2</strong></td>
<td>Student provides an equation or graph that does not reflect the correct data, OR student fails to provide an equation or graph.</td>
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<td>Student provides a correct graph or table, but the answer to the question is either not given or incorrect.</td>
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<td>Student provides a correct table or graph, but the answer is 4 months with an explanation that the 100 mark occurs during the 4th month.</td>
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<td>Student provides a correct table or graph, AND the answer is correct (5 months) with a valid explanation.</td>
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<tr>
<td>e</td>
<td><strong>F-BF.A.1a</strong></td>
<td>Student does not provide a linear function.</td>
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<td>Student provides a function is linear but does not reflect data.</td>
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<td>Student provides a correct linear function, but the function was either simplified incorrectly or does not use the notation, ( t(n) ).</td>
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<td>Student provides a correct linear function using the notation, ( t(n) ).</td>
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<tr>
<td>f</td>
<td><strong>A-CED.A.2</strong></td>
<td>Student does not demonstrate an ability to recognize and distinguish between linear and exponential growth or to compare growth rates or average rate of change of functions.</td>
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<td></td>
<td>Student makes a partially correct but incomplete comparison of growth rates that does not include or incorrectly applies the concept of average rate of change.</td>
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<td></td>
<td>Student makes a correct comparison of growth rates that includes an analysis of the rate of change of each function. However, student’s communication contains minor errors or misuse of mathematical terms.</td>
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<td>Student identifies that the Guppies’ population will increase at a faster rate and provides a valid explanation that includes an analysis of the rate of change of each function.</td>
<td></td>
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<tr>
<td>g</td>
<td><strong>A-REI.D.11</strong></td>
<td>Student does not provide correct graphs of the functions and is unable to provide an answer that is based on reasoning.</td>
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<td>Student provides correct graphs but is unable to arrive at a correct answer from the graphs, OR student’s graphs are incomplete or incorrect, but the student arrives</td>
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<tr>
<td>h</td>
<td>F-IF.B.6 F-LE.A.1 F-LE.A.3</td>
<td>at an answer based on sound reasoning.</td>
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<tr>
<td></td>
<td>Student does not provide tables or graphs that are accurate enough to support an answer, and shows little reasoning in an explanation.</td>
<td>Student provides tables or graphs that are correct but provides limited or incorrect explanation of results.</td>
<td>Student provides tables or graphs that are correct and gives an explanation that is predominantly correct but contains minor errors or omissions in the explanation.</td>
<td></td>
</tr>
<tr>
<td>i</td>
<td>A-SSE.B.3c</td>
<td>Student does not provide an exponential function that shows percent increase.</td>
<td>Student writes an exponential function that uses an incorrect version of the growth factor, such as 0.02, 2%, 20%, or 0.20.</td>
<td>Student creates a correct version of the function using a growth factor expressed as 200% or expressed as 2 with a note that 2 is equivalent to 200%. Student has a minor error in notation, or in the domain, or does not specify the domain.</td>
</tr>
<tr>
<td>a – c</td>
<td>F-BF.A.1a F-IF.A.3 F-LE.A.1 F-LE.A.2</td>
<td>Student does not fill in the table correctly and does not describe the relationship correctly. Student does not provide an exponential function.</td>
<td>Student completes the table correctly and describes the sequence correctly but gives an incorrect function. Student may or may not have given a correct domain.</td>
<td>Student completes the table correctly, and describes the sequence correctly, but has a minor error in either his or her function or domain. The function provided is exponential with a growth factor of 3. Description or notation may contain minor errors.</td>
</tr>
<tr>
<td>d</td>
<td>F-BF.A.1a F-LE.A.1 F-LE.A.2</td>
<td>Student fails to provide an explicit exponential formula.</td>
<td>Student provides an explicit formula that is exponential but incorrect; supporting work is missing or reflects limited reasoning about the problem.</td>
<td>Student provides a correct explicit exponential formula. Notation or supporting work may contain minor errors.</td>
</tr>
<tr>
<td>e</td>
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</tbody>
</table>
### End-of-Module Assessment Task

**M3**

**ALGEBRA I**

<table>
<thead>
<tr>
<th>f</th>
<th>F-BF.A.1a</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>F-LE.A.1</td>
</tr>
</tbody>
</table>

**Module 3: Linear and Exponential Functions**

**Date:** 9/12/13

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**f**

Student provides little or no evidence of understanding how to determine the perimeter of the dark triangles nor how to recognize the common factor between two successive figures’ perimeter.

**g**

Student’s value of \( k \) is incorrect or not provided, but solution shows some understanding of how to determine the perimeter of the dark triangles.

**h**

Student’s solution shows significant progress towards identifying that \( k \) is 3/2 but contains minor errors or is not complete. OR student computes an incorrect \( k \) value due to a minor error but otherwise demonstrates a way to determine \( k \) either by recognizing that the given equation is a recursive form of a geometric sequence or by approaching the problem algebraically.

**i**

Student identifies the correct value of \( k \) with enough supporting evidence of student thinking (correct table, graph, marking on diagram, or calculations) that shows how her or she arrived at the solution.

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<table>
<thead>
<tr>
<th>5</th>
<th>a</th>
<th>F-BF.A.1a</th>
</tr>
</thead>
</table>

**5**

Student does not provide a piecewise definition of the function and/or more than two expressions in the answer are incorrect.

**5**

Student provides a piecewise function in which at least one of the expressions is correct, the solution may contain errors with the intervals or notation.

**5**

Student provides a piecewise function with correct expressions, but the answer may contain minor errors with the intervals or use of function notation. OR one expression is incorrect, but intervals and use of function notation is correct.

**5**

Student provides a correctly defined piecewise function with correct intervals.

---

<table>
<thead>
<tr>
<th>b – c</th>
<th>F-BF.B.3</th>
</tr>
</thead>
</table>

**b – c**

Student’s graphs contain major errors; domain and range are missing or are inconsistent with the graphs.

**b – c**

Student’s graph for (b) would be correct for (c) and vice versa, OR student answers either (b) or (c) correctly. Minor errors may exist in the domain and range.

**b – c**

Student’s graphs contain one minor error. The domain and range are consistent with the graphs.

**b – c**

Student provides correct graphs for both (b) and (c) and provides a domain and range for each that are consistent with student graphs.

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<table>
<thead>
<tr>
<th>d – e</th>
<th>F-BF.B.3</th>
</tr>
</thead>
</table>

**d – e**

Both explanations and solutions are incorrect or have major conceptual errors (e.g., confusing domain and range).

**d – e**

Student answers contain more than one minor error, OR student answers only one of (d) and (e) correctly.

**d – e**

Student answer only explains how the domain/range changes; it may contain one minor error.

**d – e**

Student answer not only explains how the domain/range changes, but also explains how knowing \( k > 1 \) aids in finding the new domain/range.
1. Given \( h(x) = |x + 2| - 3 \) and \( g(x) = -|x| + 4 \).
   a. Describe how to obtain the graph of \( g \) from the graph of \( a(x) = |x| \) using transformations.

   \[ \text{To obtain the graph of } g, \text{ reflect the graph of } a \text{ about the } x\text{-axis and translate this graph up 4 units.} \]

   b. Describe how to obtain the graph of \( h \) from the graph of \( a(x) = |x| \) using transformations.

   \[ \text{To obtain the graph of } h, \text{ translate the graph of } a \text{ 2 units to the left and 3 units down.} \]

   c. Sketch the graphs of \( h(x) \) and \( g(x) \) on the same coordinate plane.

   \[ y = h(x) \]

   \[ y = g(x) \]

   d. Use your graphs to estimate the solutions to the equation:

   \[ |x + 2| - 3 = -|x| + 4 \]

   Explain how you got your answer.

   \[ \text{Solution: } x \approx 2.5 \text{ or } x \approx -4.5 \]

   The solutions are the \( x \)-coordinates of the intersection points of the graphs of \( g \) and \( h \).

   e. Were your estimations you made in part (d) correct? If yes, how do you know? If not explain why not.

   \[ \text{Let } x = 2.5 \]

   \[ \text{Is } |2.5 + 2| - 3 = -|2.5| + 4 \text{ true?} \]

   Yes, \( 4.5 - 3 = 2.5 + 4 \) is true.

   \[ \text{Let } x = -4.5 \]

   \[ \text{Is } |-4.5 + 2| - 3 = -|4.5| + 4 \text{ true?} \]

   Yes, \( 2.5 - 3 = -4.5 + 4 \) is true.

   Yes, the estimates are correct. They each make the equation true.
2. Let \( f \) and \( g \) be the functions given by \( f(x) = x^2 \) and \( g(x) = x|x| \).
   
   a. Find \( f\left(\frac{1}{3}\right), g(4), \) and \( g(-\sqrt{3}) \).
      
      \[
      f\left(\frac{1}{3}\right) = \frac{1}{9}, \quad g(4) = 16, \quad g(-\sqrt{3}) = -3
      \]
   
   b. What is the domain of \( f \)?
      
      \( D: \) all real numbers.
   
   c. What is the range of \( g \)?
      
      \( R: \) all real numbers.
   
   d. Evaluate \( f(-67) + g(-67) \).
      
      \[
      (-67)^2 + -67|-67| = 0.
      \]
   
   e. Compare and contrast \( f \) and \( g \). How are they alike? How are they different?
      
      When \( x \) is positive, both functions give the same value. But when \( x \) is negative, \( f \) gives the always positive value of \( x^2 \), whereas \( g \) gives a value that is the opposite of what \( f \) gives.
   
   f. Is there a value of \( x \), such that \( f(x) + g(x) = -100 \)? If so, find \( x \). If not, explain why no such value exists.
      
      No, \( f \) and \( g \) are either both zero, giving a sum of zero, both positive, giving a positive sum, or the opposite of each other, giving a sum of zero. So, there is no way to get a negative sum.
   
   g. Is there a value of \( x \) such that \( f(x) + g(x) = 50 \)? If so, find \( x \). If not, explain why no such value exists.
      
      Yes, if \( x = 5 \), \( f(x) = g(x) = 25 \), thus \( f(x) + g(x) = 50 \).
3. A boy bought 6 guppies at the beginning of the month. One month later the number of guppies in his tank had doubled. His guppy population continued to grow in this same manner. His sister bought some tetras at the same time. The table below shows the number of tetras, t, after n months have passed since they bought the fish.

<table>
<thead>
<tr>
<th>n, months</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>t, tetras</td>
<td>8</td>
<td>16</td>
<td>24</td>
<td>32</td>
</tr>
</tbody>
</table>

a. Create a function \( g \) to model the growth of the boy's guppy population, where \( g(n) \) is the number of guppies at the beginning of each month and \( n \) is the number of months that have passed since he bought the 6 guppies. What is a reasonable domain for \( g \) in this situation?

\[
g(n) = 6 \cdot 2^n \quad \text{Domain: } n \text{ is a whole number.}
\]

b. How many guppies will there be one year after he bought the 6 guppies?

\[
g(12) = 6 \cdot 2^{12} = 24,576 \text{ guppies}
\]

c. Create an equation that could be solved to determine how many months after he bought the guppies there will be 100 guppies.

\[
100 = 6 \cdot 2^n
\]

d. Use graphs or tables to approximate a solution to the equation from part c. Explain how you arrived at your estimate.
e. Create a function, \( t \), to model the growth of the sister’s tetra population, where \( t(n) \) is the number of tetras after \( n \) months have passed since she bought the tetras.

\[ t(n) = 8(n+1), \text{ if } n \text{ is a whole number.} \]

Or, \( t(n) = 8n + 8, \text{ if } n \text{ is a whole number.} \)

f. Compare the growth of the sister’s tetra population to the growth of the guppy population. Include a comparison of the average rate of change for the functions that model each population’s growth over time.

The guppies’ population is increasing faster than the tetras’ population. Each month, the number of guppies doubles, while the number of tetra’s increases by 8. The rate of change for the tetras is constant, but the rate of change for the guppies is always increasing.

g. Use graphs to estimate the number of months that will have passed when the population of guppies and tetras will be the same.

The guppies and tetras populations will be the same, 24, after 2 months.
h. Use graphs or tables to explain why the guppy population will eventually exceed the tetra population even though there were more tetras to start with.

The guppy population's growth is exponential, and the tetra populations’ growth is linear. The graph in part (g) shows how the population of the guppies eventually overtakes the population of the tetras. The table below shows that by the end of the 3rd month, there are more guppies than tetras.

<table>
<thead>
<tr>
<th>n</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>g(n)</td>
<td>6</td>
<td>12</td>
<td>24</td>
<td>48</td>
<td>96</td>
<td>192</td>
</tr>
<tr>
<td>t(n)</td>
<td>8</td>
<td>16</td>
<td>24</td>
<td>32</td>
<td>40</td>
<td>48</td>
</tr>
</tbody>
</table>

The average rate of change is doubling. The rate of change is constant.

i. Write the function \( g(n) \) in such a way that the percent increase in the number of fish per month can be identified. Circle or underline the expression representing percent increase in number of fish per month.

\[
g(n) = 6(200\%)^n
\]
4. Regard the solid dark equilateral triangle as figure 0. Then, the first figure in this sequence is the one composed of three dark triangles, the second figure is the one composed of nine dark triangles, and so on.

![Figures 0 to 4](image)

a. How many dark triangles are in each figure? Make a table to show this data.

<table>
<thead>
<tr>
<th>n (Figure Number)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>T (# of dark triangles)</td>
<td>1</td>
<td>3</td>
<td>9</td>
<td>27</td>
<td>81</td>
</tr>
</tbody>
</table>

b. Describe in words how, given the number of dark triangles in a figure, to determine the number of dark triangles in the next figure.

The number of triangles in a figure is 3 times the number of triangles in the previous figure.

c. Create a function that models this sequence. What is the domain of this function?

\[ T(n) = 3^n, \quad D: n \text{ is a whole number.} \]

d. Suppose the area of the solid dark triangle in Figure 0 is 1 square meter. The areas of one dark triangle in each figure form a sequence. Create an explicit formula that gives the area of just one of the dark triangles in the \( n \)th figure in the sequence.

\[ A(n) = \left(\frac{1}{2}\right)^n \]

<table>
<thead>
<tr>
<th>Figure, n</th>
<th>Area of one dark triangle, A(n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1/4</td>
</tr>
<tr>
<td>2</td>
<td>1/16</td>
</tr>
<tr>
<td>3</td>
<td>1/64</td>
</tr>
</tbody>
</table>
e. The sum of the areas of all the dark triangles in Figure 0 is 1 \text{ m}^2; there is only one triangle in this case. The sum of the areas of all the dark triangles in Figure 1 is \( \frac{3}{4} \text{ m}^2 \). What is the sum of the areas of all the dark triangles in the \( n \)\textsuperscript{th} figure in the sequence? Is this total area increasing or decreasing as \( n \) increases?

<table>
<thead>
<tr>
<th>Figure</th>
<th>Area in m(^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{3}{4} )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{9}{16} )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{27}{64} )</td>
</tr>
</tbody>
</table>

\[ T(n) = \left( \frac{3}{4} \right)^n \]

The total area is decreasing as \( n \) increases.

f. Let \( P(n) \) be the sum of the perimeters of the all dark triangles in the \( n \)\textsuperscript{th} figure in the sequence of figures. There is a real number \( k \) so that:

\[ P(n + 1) = kP(n) \]

is true for each positive whole number \( n \). What is the value of \( k \)?

Let \( x \) represent the number of meters long of one side of the triangle in Figure 0.

<table>
<thead>
<tr>
<th>Figure</th>
<th>( P(n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3( x )</td>
</tr>
<tr>
<td>1</td>
<td>( 3\frac{3}{2}x = \frac{9}{2}x )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{9}{2}x + \frac{9}{4}x = \frac{27}{4}x )</td>
</tr>
</tbody>
</table>

\( P \) is a geometric sequence and \( k \) is the ratio between any term and the previous term, so \( k = \frac{P(n+1)}{P(n)} \).

So, for example, for \( n = 0 \),

\[ k = \frac{P(1)}{P(0)} = \frac{\frac{9}{2}x}{3x} = \frac{3}{2} \]

For \( n = 1 \), \( k = \frac{P(2)}{P(1)} = \frac{\frac{27}{4}x}{\frac{9}{2}x} = \frac{3}{2} \)

\( k = \frac{3}{2} \)
5. The graph of a piecewise-defined function $f$ is shown to the right. The domain of $f$ is $-3 \leq x \leq 3$.

a. Create an algebraic representation for $f$. Assume that the graph of $f$ is composed of straight line segments.

$$f(x) = \begin{cases} 
  x, & -3 \leq x < -1 \\
  -1, & -1 \leq x < 0 \\
  x - 1, & 0 \leq x < 2 \\
  -x + 3, & 2 \leq x \leq 3
\end{cases}$$

or  

$$f(x) = \begin{cases} 
  x, & -3 \leq x < -1 \\
  -1, & -1 \leq x < 0 \\
  -|x - 2| + 1, & 0 \leq x \leq 3
\end{cases}$$

b. Sketch the graph of $y = 2f(x)$ and state the domain and range.

Domain: $-3 \leq x \leq 3$

Range: $-6 \leq y \leq 2$
c. Sketch the graph of \( y = f(2x) \) and state the domain and range.

\[ \text{Domain: } -1.5 \leq x \leq 1.5 \]
\[ \text{Range: } -3 \leq y \leq 1 \]

\[ y = f(2x) \]

\[ -6 \quad -5 \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \]

\[ 0 \quad 2 \quad 4 \quad 6 \quad 8 \]


d. How does the range of \( y = f(x) \) compare to the range of \( y = kf(x) \), where \( k > 1 \)?

Every value in the range of \( y = f(x) \) would be multiplied by \( k \). Since \( k > 1 \) we can represent this by multiplying the compound inequality that gives the range of \( y = f(x) \) by \( k \), giving \(-3k \leq y \leq k\).

e. How does the domain of \( y = f(x) \) compare to the domain of \( y = f(kx) \), where \( k > 1 \)?

Every value in the domain of \( y = f(x) \) would be divided by \( k \). Since \( k > 1 \) we can represent this by multiplying the compound inequality that gives the domain of \( y = f(x) \) by \( 1/k \), giving \(-3/k \leq x \leq 3/k\).