Consultation Draft

FOUNDATIONS *for* Success

Mathematics Expectations for the Middle Grades

Achieve, Inc.

MATHEMATICS ACHIEVEMENT PARTNERSHIP

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Achieve is an independent, bipartisan, nonprofit organization created by governors and corporate leaders to help raise standards and performance in American schools. Achieve was founded at the 1996 National Education Summit and subsequently sponsored additional Summits in 1999 and 2001 that brought together more than 100 governors, business leaders and educators from around the nation.

Achieve's principal purposes are to:

- provide sustained public leadership and advocacy for the movement to raise standards and improve student performance;
- help states benchmark their standards, assessments and accountability systems against the best in the country and the world;
- build partnerships that allow states to work together to improve teaching and learning and raise student achievement; and
- serve as a national clearinghouse on education standards and school reform.

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Achieve's Mathematics Achievement Partnership

Since the appearance of *Sputnik* in 1957, national leaders and educators have focused on the importance of helping U.S. students stay competitive internationally in mathematics and science. Today, Achieve's Mathematics Achievement Partnership (MAP) is a comprehensive initiative dedicated to helping states improve their middle-school students' mathematics performance — so U.S. students can measure up to the best in the world. Unique because it is based in lessons from top-performing countries, MAP is the most ambitious effort to date to create an integrated system that enables states to compare achievement and to provide schools with access to world-class training and teaching materials.

At MAP's core is *Foundations for Success*, a set of challenging content expectations for the end of grade 8 that have been benchmarked to the best international and state standards in order to reflect the core mathematical knowledge and skills students will need for success in high school and beyond.

In partnership with other national organizations, MAP will build off *Foundations for Success* to provide states with concrete tools they can use to improve mathematics instruction and student achievement, including aligned materials and tools for high quality; sustained professional development for teachers; diagnostic tests for students coupled with guidance for teachers on how to use assessment results to improve and target instruction; and a common, annual, internationally benchmarked eighth-grade mathematics test whose results — reported at both the school and student levels — will be comparable across state lines.

At the time of this printing, 14 states are collaborating with Achieve on this initiative. By working together, Achieve and its partner states are pooling resources, information and energy to affect dramatic gains in middle-school mathematics. The result will be U.S. students who are equipped to compete with the best in the world.

MAP Partner States

California	New Hampshire
Georgia	North Carolina
Illinois	Ohio
Indiana	Oregon
Maryland	Vermont
Massachusetts	Washington
Michigan	Wisconsin

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Acknowledgments

Foundations for Success is the result of the hard work of numerous people. First and foremost, we acknowledge with gratitude the members of our Mathematics Advisory Panel, whose time, energy and agile minds were invaluable to this project. We owe special thanks to Ralph Raimi, William Schmidt and Lynn Steen for their efforts that extended far beyond the call of duty.

We also would like to acknowledge individuals at the National Center for Education and the Economy for their role in helping to craft the mathematics expectations: Phil Daro, director of assessment and development, facilitated the meetings of the Mathematics Advisory Panel; and Ann Shannon, director of mathematics, worked with the panel to draft the expectations and sample problems. We thank them for their hard work, enthusiasm and patience.

We also would like to recognize a number of individuals who provided Achieve with expert advice, editorial assistance and/or moral support during the development of these expectations: Nancy Belden of Belden Russonello & Stewart; Kathy Delaney, Bill Porter and Andrea Sussman of KSA-Plus Communications; Joseph Garcia from Communication*Works*; Mary Schaefer, mathematical consultant; Celia Stevenson, graphic designer; and Malcolm Swan from the University of Nottingham in Nottingham, England. We are particularly grateful for the help of Washington-based education writer Nancy Henderson, who provided significant writing and editorial assistance.

Finally, we extend special thanks to the Achieve staff members who contributed to the creation of these expections, as well as to the broader Mathematics Achievement Partnership. Laura McGiffert, the principal manager of this project, assumed primary responsibility for the development of *Foundations for Success*; Robert Rothman provided editorial assistance; Jennifer Vranek contributed to the initial research; Mark Faust supplied additional research; and Vickie Barber and Levon Saunders provided logistical support.

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The Annenberg Foundation
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AT&T Foundation
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The Prudential Foundation
State Farm Insurance Companies
State Street Foundation
The UPS Foundation
The Washington Mutual Foundation
The William and Flora Hewlett Foundation
Williams
The Xerox Foundation

* Provided special funding for the MAP Initiative

Achieve's Mathematics Advisory Panel

The development of *Foundations for Success* was led by Achieve's Mathematics Advisory Panel, an expert panel of mathematicians, mathematics educators, curriculum specialists, and state and local mathematics supervisors representing a broad spectrum of perspectives about mathematics education. This advisory panel reviewed an analysis of standards from numerous states and countries and, after considerable discussion, made judgments about what mathematics in the middle grades should entail. After vigorous debate and careful compromise, the panel concluded that the set of expectations included in *Foundations for Success* represents the core knowledge and skills that students should learn to be prepared for high school and beyond.

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Introduction

From using a Palm Pilot to managing a retirement portfolio, life in the 21st century is drenched in data, dominated by computers and controlled by quantitative information. The daily news is full of statistics, graphs and percentages; political polls, election outcomes and census counts are based on mathematical concepts such as ratio, probability and sampling. Virtually all workers — custodians, secretaries, farmers, electricians, building contractors, store managers, architects, lawyers, graphic designers and others — need to deal with quantitative concepts and use reasoning skills in their jobs. Indeed, those who lack mathematical know-how face a very real threat of being left behind — as citizens, as consumers and as workers.

Traditionally, it has been sufficient for only some of us to be mathematically proficient. To fully participate in the 21st century, however, *all* citizens must be comfortable and fluent with mathematics. In addition to dealing with quantitative ideas in virtually any job, today's students will need mathematics to handle their finances, evaluate medical risks and understand public policy issues — from the future of Social Security to the risks of genetically modified foods. As the National Research Council (NRC) concludes in its report *Adding It Up: Helping Children Learn Mathematics*, "All young Americans must learn to think mathematically, and they must think mathematically to learn."

While U.S. students do reasonably well at straightforward calculations, numerous state, national and international assessments conducted over the past 30 years indicate that far too few have a firm understanding of fundamental mathematical concepts. Additionally, few students can use mathematics to solve straightforward, real-life problems. These shortcomings limit individual opportunity and threaten the nation's future.

To confront those challenges, Achieve, Inc.'s Mathematics Achievement Partnership (MAP) has been working with its partner states to strengthen U.S. mathematics education. MAP recognizes that improving student performance depends on a comprehensive approach based on:

- supporting teachers by equipping them with the knowledge and skills they need to help raise student proficiency;
- measuring student proficiency on a regular basis; and
- using assessment results to assist teachers and improve classroom practice.

The first step in this approach is to identify the knowledge and skills that students need to develop to be successful. *Foundations of Success: Mathematics Expectations for the Middle Grades* does just that.

These expectations are the outcome of extensive efforts by Achieve's Mathematics Advisory Panel, a diverse group of classroom teachers, curriculum specialists, state and local supervisors of mathematics education, prominent university mathematicians, and mathematics educators. Representing a wide spectrum of perspectives about mathematics education, members of the Mathematics Advisory Panel have reached general agreement on the knowledge, understanding and skills that students need to have before entering high school. These unified standards provide a common target for states and establish a strong foundation for the mathematical proficiency called for by the NRC and other leading national organizations.

Building a New Foundation

MAP's work is grounded in the 1995 Third International Mathematics and Science Study (TIMSS). This study and its 1999 follow-up, TIMSS-R, present a sobering picture of how U.S. students perform when compared with their peers around the world. While U.S. fourth graders are among the best prepared in mathematics, American student achievement rapidly falls off in the middle grades, trailing countries such as Canada, England, Australia, the Czech Republic, Japan and South Korea. By the end of high school, U.S. students do even worse, performing near the bottom internationally.

Students will need mathematics to handle their finances, evaluate medical risks and understand public policy issues — from the future of Social Security to the risks of genetically modified foods.

TIMSS data show that in too many U.S. classrooms, mathematics curricula in grades 6 through 8 simply repeat previously taught concepts and do not provide deep study in any area. The data also show that American mathematics curricula for the middle grades generally do little to advance mathematics knowledge beyond arithmetic computation. Meanwhile, students in other countries are mastering arithmetic concepts by the end of sixth grade and advancing to the fundamentals of algebra and geometry in the seventh and eighth grades. Thus, a far greater proportion of students in other countries are preparing for substantial mathematics courses in high school, while too many U.S. students are getting left behind.

Achieve reaffirmed the TIMSS findings with its own analysis of 21 state tests of fourth- and eighth-grade students. It found that more than 60 percent of the eighth-grade test items dealt with computation, whole numbers and fractions — procedures that students in other countries master before the seventh grade. In top-performing nations, seventh- and eighth-grade curricula include proportionality and slope, congruence and similarity, equations and functions, and two- and three-dimensional geometry — topics that most U.S. state tests address sparingly, if at all.

Fortunately, the United States does not have to look far to find a road map for improvement. In 1989, the National Council of Teachers of Mathematics (NCTM) initiated the drive toward higher standards in mathematics education by publishing K–12 standards that showed the breadth and depth of mathematics proficiency required for the 21st century. These standards, and NCTM's revised version, *Principles and Standards for School Mathematics*, published in 2000, laid the groundwork for MAP. These documents, and the Achieve TIMSS analysis, make the case that strong performance in mathematics requires an emphasis on procedural skills, conceptual knowledge and problem solving.

Following the TIMSS analysis, MAP asked mathematicians and mathematics educators to take a fresh look at the mathematics expectations for the middle grades. MAP blends its advisory panel's conclusions and the TIMSS findings in *Foundations for Success*, a blueprint for mathematics in the middle grades that is benchmarked to international standards. This set of expectations is designed to be challenging yet realistic, and eventually attainable by all students and teachers who are given adequate support. With the goal of providing all students with a strong foundation in mathematics before they begin high school, it aims to cultivate every student's potential. Those who master fundamental concepts in the middle grades will have the tools they need to succeed in high school, college and the workplace.

A Coordinated Approach

MAP is creating tools to help states build an enduring foundation for mathematics education: aligned expectations, professional development, curriculum materials and assessments. MAP's comprehensive approach is more than just "road repair." MAP offers a bold, forward-looking infrastructure for strengthening and updating mathematics education in the middle grades. These tools will provide students with the understanding and skills they need to succeed in high school and beyond.

1. **World-class expectations.** *Foundations for Success* is a set of ambitious expectations for the end of grade 8. It incorporates the fundamentals that students in top-performing countries are learning and the skills that U.S. students will need to succeed. It is a blueprint for reorienting mathematics in the middle grades.
2. **High-quality teacher support and curriculum guidance.** To ensure that teachers are fully prepared to help students meet world-class standards, MAP will work with states to provide teachers with extensive opportunities to improve their mathematics knowledge and teaching practices. In addition, MAP will identify materials such as sample teaching guides, ideas for study units and recommended textbooks that are aligned with the mathematics skills and understanding described in *Foundations for Success*.
3. **Ongoing diagnostic tests and an end-of-eighth-grade assessment.** These two types of tests, aligned with MAP's standards, will reflect the substance and scope of what students in the middle grades are learning in top-performing countries. Teachers will use the diagnostic tests in the classroom to track student progress in grades 6 through 8 and adjust instruction accordingly. The end-of-eighth-grade test will be offered each year as part of each state's assessment system.
   In addition, it will provide results for individual schools and students, allowing parents, educators and policymakers to compare student achievement across state lines. That is not possible with existing state tests because different tests measure different skills and knowledge.

Setting World-Class Expectations

Foundations for Success offers guidelines and targets for states to provide mathematics education that is benchmarked to the best in the world. It identifies the skills and knowledge that will underlie MAP's professional development, curriculum and assessment tools.

The expert panel of mathematicians, mathematics educators, classroom teachers and curriculum specialists who have worked with Achieve to develop these expectations represents a wide spectrum of perspectives about mathematics education (Appendix A). Rich with algebra, geometry and data analysis, *Foundations for Success* represents a balanced and informed view of the necessary emphases and scope of mathematics in the middle grades. Students who have met these expectations will have strong computational and reasoning skills, the ability to work with abstract ideas and complex situations, and the ability to interpret data and solve real-life problems.

To help illustrate what the MAP expectations mean, a number of sample problems and methods for solving these problems accompany the outline of learning objectives. These problems do more than show procedures and skills. They demonstrate the depth of mathematical understanding and reasoning skills implied by the expectations. The primary purpose of these problems is to assist curriculum developers and teacher educators as they rethink their learning objectives for students in the middle grades. Teachers also may find them useful for classroom discussion. The accompanying solutions do not represent sample student-produced solutions, and MAP does not intend that schools use the illustrative problems to assess student performance.

MAP recognizes that *Foundations for Success* encompasses more mathematics than most U.S. students are currently learning by the end of the eighth grade. In fact, it covers more than some students now learn by the time they finish high school. There is no doubt, however, that MAP's expectations are realistic goals for any 14-year-old who is provided with adequate support, including strong preparation during grades K–5.

These standards will not be achieved overnight, but they are attainable, appropriate and necessary. With thorough professional development for teachers and comprehensive instruction for students in grades K–8, U.S. students eventually will be able to perform at the same level as their peers around the world.

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Frequently Asked Questions About MAP

THE NEED FOR HIGH STANDARDS

Why is it so important for all students to become mathematically proficient?

For life in the 21st century, mathematical proficiency is as basic as literacy. Mathematics permeates society and the economy like never before. Mathematical models undergird robotics, genetics research, computer graphics, cryptography, and countless other scientific and technological endeavors. From marketing to construction and from agriculture to manufacturing, data rule decisionmaking. In addition to dealing with quantitative ideas in virtually any job, individuals confront mathematical and statistical concepts in managing personal finances, evaluating medical risks and interpreting public policy issues. Concepts such as exponential growth and margins of error are now as important to the individual as the more basic arithmetic of calculating discounts or tips. The uncertainty about vote counts in the recent presidential election is a prime example of how dealing with data is no simple matter. To participate fully in a 21st century democracy, all students need to become mathematically proficient.

What is mathematical proficiency?

Proficiency requires much more than just "the basics." It involves a blend of basic knowledge with conceptual understanding, together with the ability to solve mathematical problems. Proficiency encompasses reasoning ability, logical thinking, problem solving and procedural skills. In its recent report *Adding It Up: Helping Children Learn Mathematics*, the NRC identifies five interrelated aspects of mathematical proficiency: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition. With these characteristics, mathematical proficiency provides the ability to use mathematics in a wide variety of situations — in school, at home and on the job.

What's new and different about MAP's expectations?

Foundations for Success *builds on the widely influential standards developed by the NCTM, as well as a number of state standards documents.* MAP's lens is, however, somewhat different: MAP focuses on what countries with top-performing students expect in the middle grades.

Educators will see many areas of overlap between *Foundations for Success*, NCTM's *Principles and Standards for School Mathematics* and current state standards, but they also will see important distinctions. In particular, MAP focuses on what has given other countries an advantage in the middle grades — fundamentals. Moreover, the MAP expectations generally represent a higher level of data analysis, geometry and algebra than most states require by the end of eighth grade.

RAISING THE BAR

How much mathematics are U.S. students currently learning?

Not enough. Evidence from state, national and international studies shows that many U.S. students fail to meet even traditional goals for eighth-grade mathematics — competence in arithmetic and readiness for algebra. According to the National Assessment of

Educational Progress, fewer than 30 percent of U.S. fourth- and eighth-grade students achieve beyond the "basic" level for their grade in mathematics. By 12th grade, fewer than 20 percent achieve beyond "basic." Results from TIMSS confirm this finding: While American fourth graders placed near the top of the 40 participating nations, U.S. eighth graders placed below the international average. By the end of high school, American students were near the bottom of the performance charts.

If students already are having a difficult time meeting existing standards, isn't it unrealistic to pursue more ambitious expectations?

Students need to be prepared for the challenges they will face in college and on the job. Nearly one-third of students who enter college immediately after high school graduation enroll in remedial mathematics. In addition, U.S. employers cannot find enough skilled workers: A recent survey of more than 1,000 employers found that more than one-third of job applicants are turned away because they lack mathematical and verbal skills needed for the jobs they seek. The need for skilled workers will only increase, as the U.S. Bureau of Labor Statistics reports that the five fastest-growing types of employment through 2008 will be in fields that depend on mathematics.

MAP's expectations are realistic goals for any 14-year-old student who is provided with adequate support. How do we know this? Students in other countries routinely attain these goals. U.S. students are as capable as students anywhere, but they need the same learning opportunities as their international peers. In contrast with other nations, the United States simply has not made effective use of the first eight years of mathematics instruction.

But won't higher expectations lead to a situation where even more students will fail?

Not if students are well prepared and if schools faithfully monitor student progress, ensure that teachers are well prepared and provide appropriate intervention to make sure that students do not fall behind. MAP's approach combines high expectations with frequent feedback on student performance and professional development for teachers. Achieve realizes that it will take time for schools to implement programs that reflect MAP's expectations and for students to reach these high standards. It may take a generation to get to the point where every student succeeds, but now is the time to begin.

How will the MAP expectations and assessments affect low-income students, racial minorities, English language learners and students with disabilities?

MAP's approach will help ensure that all students receive whatever support they may need to acquire a strong foundation. The bottom line is that high expectations in mathematics will serve all students well — including those who are disadvantaged. Experience has shown that when students and teachers receive adequate support, learning follows. Because the MAP assessments will be diagnostic tools, they will help identify which students need extra support and what types of support they need. Schools then will know just what they need to do to ensure that no student is left behind.

Although performance gaps in mathematics are evident at all levels of schooling, they widen significantly in the middle grades, to the point where those who are farthest behind have little chance of catching up during the high school years. By focusing attention on the middle grades, *Foundations for Success* can help ensure that everyone enters high school well prepared for further learning.

Does *Foundations for Success* do more than just introduce algebra and geometry in earlier grades?

Yes, it differs from current practice in several important ways:

- *It places special emphasis on reasoning — the lifeblood of mathematics.* This emphasis signals to schools that there is more to mathematics than procedural knowledge — even in learning about numbers, and even in the elementary school grades. The ability to reason mathematically is the one quality that scientists, engineers and employers in the skilled trades find most lacking among high school students and graduates. The lack of reasoning capability among students is widely known. Traditionally, high school geometry has been the place where students have been introduced to rigorous thinking through formal proofs; in recent years, however, this tradition has been on the wane. *Foundations for Success* recommends not only an emphasis on geometric reasoning, but also a healthy emphasis on empirical geometric experiences. This exposure is crucially important because many of students' difficulties with abstract geometric proofs have roots in their lack of empirical background in geometry. Without empirical experience, many students have great difficulty making sense of proofs.

- *It emphasizes measurement and data analysis.* Numbers form the bridge from elementary to middle grades mathematics, not only as a stepping stone to the abstractions of algebra, but also as a pathway to the practical tools of risk analysis and statistical reasoning. Students in middle grades need to extend the hands-on experiences of elementary school arithmetic to increasingly realistic experiences with measuring and sampling. They need to gain appreciation for the inaccuracies of measurement, the uncertainties of sampling and the risks associated with drawing inferences from situations where information is only partially known. Wisely chosen examples help develop these experiences and link them with concepts in algebra and geometry.
- *It introduces the foundations of algebra in an age-appropriate way. Foundations for Success* expects students not just to learn to apply formulas, but to become fluent in the use of symbolic notation and in solving linear equations. The expectations also include quadratic functions and their related parabolic graphs, but they omit the quadratic formula itself for an important pedagogical reason: It is more important for students to focus on general procedures than on any particular formula.

The Middle Grades

Why does *Foundations for Success* focus on the middle grades?

There are several important reasons for reorienting mathematics in the middle grades:

- *The middle grades are the period when American students clearly begin to fall behind their international peers.* According to evidence from TIMSS, the drop-off in mathematics achievement among U.S. students between fourth and eighth grades reflects a weakening in curriculum and instruction. This downward slide continues, and American students rank near the bottom internationally by the end of high school. TIMSS did more than just rank students internationally. It included a curriculum study that found the mathematics curriculum in U.S. middle grades to be far less challenging and less coherent than the curriculum in other countries, especially countries with the highest levels of achievement. While other nations concentrate on fundamental mathematical subjects such as algebra and geometry in the middle grades, most U.S. students are still doing elementary arithmetic.

While American fourth graders placed near the top of the 40 participating nations, U.S. eighth graders placed below the international average. By the end of high school, American students were near the bottom of the performance charts.

- *By giving all students a solid foundation in the middle grades,* Foundations for Success *supports equity and opportunity for all.* Students who are mathematically well prepared by the end of eighth grade are much more likely to take challenging mathematics courses in high school and then go on to college. Students who meet MAP's expectations will be well prepared in the basic mathematical tools required for work and for further study, notably algebra, geometry and data analysis.
- *The middle grades are where students' minds and career options are most flexible.* While demand for U.S. scientists and engineers is rising rapidly, the number of U.S. college graduates in engineering, mathematics and computer science has been on the wane since 1986. The middle grades are the place to begin reversing this trend. To pursue a career in science or technology, students need a significant program of high school mathematics, which requires the foundations of algebra, geometry and data analysis in the middle grades. Giving U.S. students a world-class mathematics education is the only way to produce enough highly skilled U.S. workers. To strengthen middle school mathematics is to invest in the future of America.

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- *Focusing on the middle grades is a strategic first step toward improving the entire K–12 mathematics curriculum.* Even if time, money and energy were limitless, trying to revamp the entire K–12 curriculum all at once would be fraught with overwhelming difficulties. The middle grades are the crossroads where students ready themselves to acquire advanced mathematical skills and knowledge. By focusing first on the middle grades, MAP hopes not only to give students a firm foundation during a crucial period of their academic development, but also to help frame the goals for high school and elementary school mathematics. The MAP expectations set the stage for high school mathematics and give elementary schools clear indication of the kind of mathematical achievements to which they should aspire.

Why do all students need to learn algebra, geometry and data analysis in the middle grades? What's wrong with learning at a slower pace?

Nothing is wrong with students who learn at a slower pace. What's wrong are school systems that force many of their students to learn too slowly. We know from international studies and from high-achieving school districts in the United States that MAP's expectations are realistic goals for any 14-year-old student who is provided with adequate support. Students who learn algebra, geometry and data analysis in the middle grades will be able to pursue the kind of mathematics in high school that will leave them well prepared for success both in college and in their future jobs. Not every student must learn at this same pace, but all students must have the opportunity and encouragement to do so. Students who do not meet these standards by the end of eighth grade simply will have more limited opportunities in high school and beyond.

In the middle grades, it is important to teach all students as if they may pursue an ambitious high school and college curriculum. Tracking young adolescents into a curriculum that ignores their potential or puts college out of reach is unconscionable.

Supporting Teachers

Why do teachers like *Foundations for Success*?

It includes illustrative problems and sample solutions that focus on concepts that need clarification or are difficult to teach. Teachers find these examples especially helpful as they prepare for class discussions. In addition, *Foundations for Success* includes supplementary notes that call attention to potential pitfalls involving subtleties in mathematical language (Appendix C). Students and teachers can find it difficult to grasp some concepts — equality and equation, linearity and proportion, abstraction and generality — because these terms are used in confusing and contradictory ways in mainstream textbooks and in common discourse. The illustrative problems also include specific examples that help to clarify subtleties in terminology.

It offers the chance to teach more meaningful mathematics. Experienced teachers know that too often mathematics curricula in the seventh and eighth grades offer students little opportunity to advance beyond the mathematics taught in the sixth grade. Teachers who are frustrated by review, review and more review — a spiral curriculum gone flat — will be excited by the chance to teach meaningful skills and concepts. And their students will benefit from the chance to replace mind-numbing review with interesting and relevant mathematics.

What kinds of support can teachers expect?

First, they will get numerous opportunities to improve their mathematics knowledge and teaching practices. Currently, few states require teachers to have a license that is specific to the middle grades, and few require a specific mathematics background. As a result, many teachers in grades 6 through 8 do not have the mathematical knowledge that they need to teach the fundamentals of data analysis, geometry and algebra. To help teachers acquire the necessary knowledge and skills, MAP will help states provide ongoing professional development. Rather than the customary elementary introduction to advanced mathematics, however, teachers in the middle grades will receive opportunities for sustained lessons that provide a deep and rich understanding of the mathematics that students should learn before entering high school. Further-more, the leverage of coordinated action among participating states will be a powerful force in the success of their professional development programs.

Teachers also will get a wealth of useful teaching tools. MAP will identify sample teaching guides, ideas for study units, textbooks and other materials that address the mathematics skills and understanding outlined in *Foundations for Success*. Thus, in addition to professional development, teachers and schools in participating states will have access to a plethora of resources, including:

- MAP's world-class expectations;
- examples and clarifications — the illustrative problems and supplementary notes in *Foundations for Success* clarify important concepts and subtleties in mathematical language (Appendix C), so these will be useful tools for professional development;
- a "consumer guide" to curriculum materials and textbooks currently available;
- juried lessons prepared by teachers for use in professional development and in student instruction;
- diagnostic tests for classroom use; and
- an Internet-based practice version of the MAP eighth-grade assessment for teachers to use in the classroom and for parents to use with students at home.

How will teachers know that their students are meeting MAP's expectations?

Through ongoing embedded assessments and an end-of-eighth-grade test. MAP's classroom diagnostic tests will allow teachers to track students' progress throughout grades 6, 7 and 8, and adjust instruction in accordance with the results. In addition, MAP will offer an end-of-eighth-grade assessment annually, which will measure individual student performance with respect to the MAP expectations.

The K–12 Continuum

What mathematics do students need to learn in the elementary grades?

Students need to enter the middle grades with confidence that mathematics is a source of useful tools for solving interesting problems. To build confidence and enthusiasm, students need strong preparation from kindergarten to grade 5, including:

- fluency with manual computation and mental estimation;
- experience visualizing and drawing geometric objects;
- practice formulating mathematical questions from various contexts; and
- plenty of opportunities to explain and critique mathematical thinking and use mathematics to solve problems.

Elementary school students need to become fluent with the basic computations of arithmetic, and they also need to understand why these procedures are valid and what concepts they represent. Thorough understanding grows best from extensive hands-on experience — in measuring and counting, in exploring common geometric objects, and in representing data in different forms. It is not enough to focus just on computational and procedural skills, because students' ability to reason mathematically depends on a deep understanding of central mathematical concepts. In turn, procedural skills provide firm support for conceptual understanding.

No short list of topics alone can encompass all the skills and knowledge required for such deep understanding, but Appendix A lists many topics that students need to learn before beginning a program for the middle grades based on the MAP expectations. It is important that students encounter topics in a sequence that respects the inherent logic of mathematics and in ways that suit students' development of mathematical understanding.

If students enter high school already knowing the fundamentals of algebra, geometry and data analysis, what is left for them to learn?

Plenty. Whether students are heading toward college, technical training or a job after high school, they will need advanced mathematics. Following a significant curriculum in the middle grades, all students should complete three years of high school mathematics covering advanced topics in statistics, geometry and algebra. (Appendix B shows one of the many ways in which that may be accomplished.) These subjects form not only the foundation of mathematics, but also of science, economics, business, medicine and many other fields of study. By making understanding of fundamental mathematical ideas an expectation of high school entry, *Foundations for Success* will encourage high school classes in all subjects to start employing the powerful tools and concepts of mathematics. Strengthening high school students' mathematics and quantitative thinking skills will help to reduce significantly the number of students who leave school with no better option than a dead-end job.

Illustrative Problems

What is the purpose of the problems and solutions?

The primary purpose of the problems and sample solutions is to illustrate the scope, depth and meaning of the expectations. For example, an expectation that students should be able to "analyze verbal problems and generate appropriate algebraic expressions" can mean many different things — from writing $A = lw$ in order to calculate the area of a rug, to writing formulas that represent the costs of different cell phone contracts.

Whenever curricular goals are interpreted simplistically, students pay the price of lowered expectations. In calibrating the broad language of expectations to high standards, the illustrative problems and sample solutions are an integral part of *Foundations for Success.*

Why isn't each problem matched directly to a specific expectation?

Although some problems do align with particular expectations (for example, understanding square roots or greatest common divisors), most involve several expectations. These problems often cut across the major strands of number, data, geometry and algebra. This is because mathematics itself is highly interconnected, and most real problems that depend on mathematical thinking for solutions draw on several parts of mathematics. Although narrow problems designed to reinforce individual learning objectives may have a place in mathematics instruction, most of them are too limited in depth and connections to convey adequately the intentions of the MAP expectations.

By making understanding of fundamental mathematical ideas an expectation of high school entry, *Foundations for Success* will encourage high school classes in all subjects to start employing the powerful tools and concepts of mathematics.

Aren't these problems too sophisticated for middle school students?

Like the expectations themselves, the illustrative problems are intended for adults, not students. They are designed to help teachers, curriculum developers, administrators, school board members and parents understand what mathematics instruction in the middle grades should aim for. Although many of these problems could serve well as catalysts for classroom discussion, they were not written in the language and style most suitable for middle school students.

Do these problems represent the kind of questions that will be on MAP assessments?

MAP intends to provide assessments that will help teachers monitor students' progress in comprehension and performance at the depth implied by these illustrative problems. Questions that accomplish this goal in the context of a broad-scale test will undoubtedly be more limited in scope and complexity than those used here to illustrate the expectations. However, the depth of understanding required for adequate performance will be comparable.

Why do so many solutions provide elaborate technical details?

What appear to be technical details are in almost all cases logical distinctions required for correct mathematical inference. The difference between and 3.1416, while small, is of crucial difference in mathematical thinking. So is the difference between rational and irrational numbers, between equations and functions, and between ratios and percentages. An important role played by the sample solutions is to bring these distinctions to the foreground for teachers and curriculum developers.

Expectations and Sample Problems

Foundations for Success identifies the mathematical knowledge and skills that students need before entering high school. It primarily focuses on the mathematical content of the middle grades (6 through 8), with the assumption that students have completed a strong program of elementary school mathematics in grades 1 through 5 (Appendix A). Students who successfully meet these expectations for the middle grades will be well prepared for a strong mathematics program in grade 9 and beyond (Appendix B), and for high school subjects such as science and social studies that increasingly depend on mathematical skills and knowledge.

These expectations comprise four strands, each of which encompasses three primary topics (see box below).

Each strand contains a brief introduction to clarify its purpose; a summary to convey its scope; and a set of expectations concerning what students need to know, understand and be able to do in each topic area.

In addition, each strand is accompanied by a number of sample problems designed to help illustrate the scope, depth and meaning of the expectations. These problems show more than just procedures and skills. They demonstrate the depth of mathematical understanding and reasoning skills that students need in order to become engaged citizens and productive employees in the 21st century. They illustrate, at the eighth-grade level, many aspects of mathematical proficiency that the NRC identifies in its recent report *Adding It Up: Helping Children Learn Mathematics* (e.g., conceptual understanding, procedural fluency, strategic

Strands	**Topics**
I. Number	A. Whole numbers
	B. Rational numbers
	C. Real numbers
II. Data	A. Measurement and approximation
	B. Data analysis
	C. Probability
III. Geometry	A. Common figures
	B. Measurements
	C. Transformations
IV. Algebra	A. Symbols and operations
	B. Functions
	C. Equations

competence and adaptive reasoning). The Mathematics Achievement Partnership's use of the term "understand" refers to these interrelated aspects of mathematical proficiency.

MAP does not intend that schools use the illustrative problems to assess student performance. Furthermore, the accompanying solutions do not represent sample student-produced solutions. Rather, the problems and solutions are intended to assist teachers, curriculum developers and teacher educators as they rethink their learning objectives for students in the middle grades. The sample problems focus on concepts that need clarification or are difficult to teach. (Appendix C discusses some of the subtle mathematical issues confronting teachers in the middle school grades, which the illustrative problems seek to clarify. Future reports from Achieve will include sample exercises for use in professional development programs and sample problems for student assessment.)

Although *Foundations for Success* presents the middle grades expectations in the context of distinct strands and topics, it is important to recognize that the fundamental concepts in each strand of mathematics are inherently interconnected. Indeed, each strand uses tools from other strands, and each can serve to help introduce or review skills in other strands. Many of the expectations could fit naturally under two or three different strands, and many of the illustrative problems can be solved using tools from various strands. For this reason, many different curricula can be used to teach to these expectations; MAP does not intend to suggest that topics necessarily should be taught in the order in which they appear
in this document.

Because these expectations focus on the mathematical competence to be expected of students entering high school, they rarely make any reference to the role of calculators (or other pedagogical aids) in classroom instruction. With few exceptions, students should be able to carry out most calculations implied by these expectations without the aid of a calculator. Although it is appropriate for students to use calculators to learn mathematics and to help solve problems, they should be able to demonstrate understanding and competence both with and without the aid of calculators.

FOUNDATIONS *for* Success

Expectations

I. Number

Conceptual depth and computational fluency in the arithmetic of rational numbers is the backbone of mathematics in daily life and the foundation of quantitative thinking in science and business. It also is of critical importance for the further study of mathematics. Data analysis, geometry and algebra all depend on an understanding of the real number system for working with measurements, coordinate systems and functions.

Students should enter the middle grades being fluent with the standard operations of arithmetic. By the time they complete eighth grade, they need to understand the concept of irrational numbers and the real number system. They also need to be able to compute with ratios, percentages, square roots and exponents. In order to acquire a strong foundation for studying the mathematics contained in these expectations, students will need to accomplish much of the work in the number strand early in their middle school years.

A. Whole numbers

Well before the end of grade 8, students need to understand and be able to use relationships among whole numbers to solve a variety of problems. As soon as possible in the middle grades, students need to understand factors, multiples and primes, and be able to solve problems using those concepts.

Students should understand:

- that the number zero is an integer that is neither negative nor positive;
- the concepts of factor (or divisor), common factors and greatest common divisor;
- the concepts of multiple, common multiples and least common multiple;
- the concepts of prime and composite numbers; and
- that each whole number can be factored into the product of primes.

Students should be able to:

- use zero appropriately in arithmetic calculations;
- use the uniqueness of prime factorization to solve problems;
- find factors and multiples of three-digit positive integers;
- identify all two-digit prime numbers; and
- find the prime factors of any three-digit positive number.

B. Rational numbers

In order to be able to use numbers routinely and flexibly, all students need a thorough grounding in the decimal number system, the arithmetic of rational numbers and the concept of ratio.

Students should understand:

- that rational numbers are quotients of integers (positive or negative) with a nonzero denominator;
- ratio as a comparison of two quantities by division (as difference is their comparison by subtraction); and
- percentages as standardized ratios with denominators of 100.

Students should be able to:

- order rational numbers and place them on the number line;

- perform accurately manual multistep calculations involving addition, subtraction, multiplication and division of rational numbers;
- estimate results before performing computations;
- demonstrate understanding of the procedures used in computations;
- represent rational numbers as fractions or decimals and translate between these representations; and
- calculate percentages and use them to solve common problems about sales tax, tips, interest, discounts and compound interest.

C. Real numbers

In order to work comfortably with functions and graphs, students need a working knowledge of the entire real number system, its visual representation as a number line, other ways of representing rational numbers and some common examples of irrational numbers.

Students should understand:

- concepts of square root, cube root and n^{th} root;
- why it is that, when converted to decimal form, rational numbers either terminate or eventually repeat;
- that any number whose decimal expansion is either finite or repeating represents a rational number; and
- that irrational numbers are those which by definition cannot be expressed as quotients of integers.

Students should be able to:

- estimate square roots and cube roots;
- use integer exponents to express numbers in scientific notation; and
- name some common examples of irrational numbers (e.g., 8 , $\sqrt{2}$) and locate them on the number line.

II. Data

From questions of medical risk to political polls and stock market indices, numerical data often convey critical information. Gathering, measuring, counting, representing, summarizing, transforming, projecting and interpreting data are essential mathematical skills. Students' exposure to the complexity of measurement is their first step toward becoming skeptical and critical users of data.

Since data derived from measurements are approximate and uncertain, the topics of measurement, approximation, data and probability fit naturally under a single strand. They also provide important support for other strands. Probability is both derived from data and used for predicting future events; it is a natural application of ratios and provides a foundation for the study of statistics in high school and beyond. Similarly, much of the foundation needed to solve geometry problems rests on concepts of measurement.

A. Measurement and approximation

Students at all grade levels need to practice working with physical measurements and estimating quantities. In the middle grades, students need to enlarge their experience from simple physical measurements, such as length and weight, to more subtle measurements, such as speed, density, inflation and stock market indices. These indirect and derived quantities are used widely to measure physical, social, medical, political and economic phenomena. Experience with measurement forms a concrete basis for understanding concepts such as proportionality in algebra and similarity in geometry.

Students should understand:

- the relation between measurements and units (e.g., that all measurements require units and that a quantity accompanied by a unit represents a measurement);

- that most measurements are approximations;
- common metric and English units of measurement for length, area, volume, time and weight;
- indirect and derived quantities such as density, velocity and weighted averages;
- how the precision of measurement influences the accuracy of derived quantities calculated from measured quantities; and
- the role of significant digits in signaling the accuracy of measurements.

Students should be able to:

- use common measuring tools accurately and select appropriate units when measuring;
- give answers to a reasonable degree of precision in the context of a given problem;
- convert between basic units of measurement within a single measurement system (e.g., square inches to square feet) and between common measurement systems (for example, inches to centimeters);
- solve problems that involve ratio units, such as population density (persons per square mile), air pressure (pounds per square inch), and speed (miles per hour);
- calculate weighted averages such as course grades, consumer price indices and sports ratings;
- convert ratio quantities between different systems of units, as, for example, feet per second to miles per hour; and
- judge reasonableness of answers by mental estimation.

B. Data analysis

Data analysis especially is appropriate for the middle grades because it forms a natural bridge from the concrete arithmetic of elementary school to the more abstract mathematics of high school. It also provides rich opportunities to develop or refresh mathematical skills such as computation, graphing, percentages and estimation.

Students should understand:

- the uses and limitations of common graphs and charts;
- relative and cumulative frequencies and associated ratios or decimals;
- appropriate and inappropriate uses of mean and median;
- the difference between correlation and causation; and
- the relation of correlation to the estimation of the line of best fit in a scatter plot.

Students should be able to:

- collect, organize and analyze both single-variable and two-variable data;
- represent and interpret data using a variety of graphs and charts, including box plots and stem-and-leaf plots;
- calculate relative and cumulative frequencies;
- find and interpret the median, upper quartile, lower quartile and inner-quartile range of a set of data;
- interpret main features of the graph of the normal distribution; and
- create and interpret scatter plots, visually estimating correlation and lines of best fit.

C. Probability

Beginning in early childhood, students' engagement with games and sports creates intuitive ideas about probability, some correct and some not. In the middle grades, students need to begin more systematic study of probability, both as preparation for living intelligently in a world of risks and as a foundation for high school and college courses in science and statistics. The relation between probability and ratio provides an ideal setting to begin more structured analysis that also has broad links with data analysis.

Students should understand:

- the relation of probability to relative frequency;
- different ways of expressing probabilities (e.g., as decimals, percentages, odds);
- why the probability of an event is a number between zero and one; and
- common misconceptions about probabilities associated with dependent and independent events; (e.g., lotteries, "hot streaks" in sports).

Students should be able to:

- solve simple problems involving probability and relative frequency;
- compare the probability of two or more events and recognize when certain events are equally likely; and
- compute probabilities of events from simple experiments with equally probable outcomes (e.g., tossing dice, flipping coins, spinning spinners).

III. Geometry

Since ancient times, geometry has been an integral part of mathematics and a common vehicle for teaching deductive reasoning, which is a defining characteristic of mathematics. Geometry provides the foundation for many common uses of mathematics, from architecture and manufacturing to computer graphics and telecommunications.

In the middle grades, students need to move beyond just describing shapes to gain an understanding of quantitative geometric relationships in two- and three-dimensional space. They also need to recognize the many connections between geometry and various concepts of number, measurement and algebra. Indeed, measurement is integral to geometry (the root meaning of which is "measuring the earth"). Moreover, similar right triangles underlie the definition of slope and play a role in many problems involving proportionality.

A. Geometric figures

Both before and during the middle grades, students need to become familiar with common types of two- and three-dimensional figures. They need to learn basic facts about lines, angles, circles and spheres; and understand basic properties of common triangles, quadrilaterals, cubes, prisms, cylinders, pyramids and cones. Students also need to be able to use their knowledge of two- and three-dimensional figures to solve problems.

For two-dimensional figures, students should understand:

- that angles around a point add to 360° and angles on one side of a line add to 180°;
- that the sum of the interior angles of a triangle is 180°;
- the triangle inequality;
- the relationships of vertical (opposite), adjacent and supplementary angles;
- that if a line intersects two parallel lines, then the corresponding angles and the alternate interior angles are equal;

- that, conversely, if a line intersects two other lines and the corresponding or alternate interior angles are equal, then the two lines are parallel;
- that polygons can be divided into triangles, which can be used to find areas, angles and sums of interior angles;
- that the sum of the exterior angles of a polygon is 360°;
- that a triangle inscribed on the diameter of a circle is a right triangle; and
- that a tangent to a circle forms a right angle with the diameter at the point of tangency.

Students should be able to:

- work flexibly with common types of two- and three-dimensional figures;
- prove the Pythagorean theorem using an area dissection argument; and
- use their understanding of lines, angles, circles, triangles and simple quadrilaterals to solve reasoning problems that involve these figures.

For three-dimensional figures, students should understand:

- properties of spheres (diameters, cross sections and great circles);
- properties and common examples of cylinders (cubes, circular cylinders and prisms);
- properties and common examples of cones (circular cones and pyramids); and
- the names and characteristics of other special solids (e.g., regular polyhedra).

Students should be able to:

- solve problems that require knowledge of common solids;
- describe the shapes of two-dimensional sections that result when a cube, cylinder, cone or sphere is cut with a plane at various angles;
- sketch a variety of two-dimensional representations of three-dimensional solids; for example, orthogonal views (top, front and side), picture views (projective or isometric) and nets (plane figures that can be folded to form the surface of the solid); and
- use two-dimensional representations to solve problems (for example, use nets to calculate surface area).

B. Measurement

Calculating lengths, areas and volumes is a fundamental mathematical skill students learn in the middle grades and apply in numerous venues thereafter. Carrying out geometric calculations provides a useful application (and review) of arithmetic. Moreover, the formulas for geometrical measurements provide important examples of functions and equations studied in algebra.

Students should understand:

- the concepts of length, area, volume and surface area;
- the relationships among one-, two- and three-dimensional units of measurement;
- how to represent and calculate areas for triangles, quadrilaterals, circles and other shapes built from these basic forms; and
- the formulas for volumes for common solids such as cylinders, cones and hemispheres:
  - a cylinder of height h whose base has area A, $V = Ah$;
  - a cone of height h whose base has area A, $V = (\frac{1}{3})Ah$; and
  - a hemisphere of height h whose base has area A, $V = (\frac{2}{3})Ah$.

Students should be able to:

- find lengths of line segments in geometric figures in terms of other given lengths;
- find the length of a circular arc in terms of its radius and central angle;
- use the Pythagorean theorem and its converse to solve perimeter, area and volume problems and to find distances between points in the Cartesian coordinate system;

- solve problems involving areas of triangles, quadrilaterals and circles;
- find the surface area of prisms, cylinders and other rectangular solids; and
- solve problems involving the volumes of cylinders, cones and spheres.

C. Transformations

The study of shape and size comes together in dealing with ideas of congruency, similarity and transformations of geometric objects. To be able to make use of geometry as a tool for solving problems, students need to be familiar with the consequences of transforming geometric figures by translation, rotation, reflection, expansion and contraction.

Students should understand:

- that two-dimensional figures having the same shape and size are congruent;
- that two-dimensional figures having the same shape are similar;
- the effects of reflecting, rotating, translating, expanding and contracting simple two-dimensional figures; and
- that in similar figures, the ratios of corresponding lengths are equal and the corresponding angles have equal measures.

Students should be able to:

- use similar triangles to measure distances indirectly;
- investigate symmetries of two- and three-dimensional figures;
- determine from side and angle conditions when triangles are similar or congruent;
- find the lengths of sides in a figure, given the scale factor and dimensions of a similar figure; and
- create, interpret and use scale drawings to help solve problems.

IV. Algebra

The middle grades provide a crucial and challenging transition from the concrete mathematics of elementary school to the more abstract and conceptual mathematics of high school. Nowhere is this challenge more striking or more important than in the transition from arithmetic to algebra — from calculating with numbers to using letters, which at first represent numbers and are later treated as entities in their own right.

In the middle grades, students learn to represent mathematical relationships symbolically and to translate real-world relationships into algebraic expressions, equations or inequalities. They learn that the familiar rules of arithmetic remain valid when letters are used to represent numbers and that these rules can be used to simplify algebraic expressions and solve algebraic equations. The basic tools of algebra provide the necessary underpinnings for high school science, social studies and mathematics.

In addition to using symbols that represent quantities, algebra introduces the concepts of functions and equations to represent relationships. Students need to learn to think about formulas such as $A = \quad r^2$ as representing a functional relationship between two quantities. For understanding functional relationships, facility with graphing is as important as working with algebraic expressions. By the end of the eighth grade, students should be able to translate among symbolic, graphic, numerical and verbal representations of functions. They should also be able to analyze algebraically the properties of lines, circles and triangles. In the middle grades, students should study linear equations in depth to the point of mastery. Students also need an introduction to common nonlinear functions, especially quadratics, as a foundation for further study in high school.

A. Symbols and operations

In the middle grades, students need to learn to use letters as symbols for variables, just as they have been using numbers in arithmetic. In particular they need to learn to use variables flexibly, to manipulate algebraic expressions accurately and to generate their own algebraic expressions to represent real-world problems.

Students should understand:

- the various contexts in which letters are used to represent numbers — to represent an unknown value, a placeholder for something not yet known or a constant;
- the conventions for writing algebraic expressions (e.g., omitting the symbol for multiplication and writing numbers before letters in multiplication); and
- the meanings of constant, variable and parameter, and relations among them.

Students should be able to:

- use variables to represent the value of quantities in a variety of contexts;
- analyze verbal problems and generate appropriate algebraic expressions;
- use properties of the real number system (e.g., distributive and associative laws) to simplify expressions; and
- recognize, represent geometrically and apply the common formulas:
  $(a + b)^2 = a^2 + 2ab + b^2$;
  $(a - b)^2 = a^2 - 2ab + b^2$; and
  $(a + b)(a - b) = a^2 - b^2$.

B. Functions

Functions are fundamental to mathematics and need to be studied in conjunction with equations. It is important, however, to distinguish functions from equations. In the middle grades, students primarily need to gain experience with linear functions in order to work with them routinely and flexibly. In these grades, students also need to become familiar with simple examples of some common nonlinear functions.

For linear functions, students should understand:

- that linear functions are characterized by constant rates of change;
- that in a graph of $y = kx + b$, the slope k is the rate of change and the parameter b is the value of y when $x = 0$; and
- that a proportional relationship $y = kx$ is a special type of linear function in which the parameter $b = 0$ and k is the constant of proportionality.

Students should be able to:

- recognize whether information given in a table, graph or formula suggests a proportional, linear or nonlinear relationship;
- represent linear functions using verbal descriptions, tables, graphs and formulas, and translate between these representations;
- recognize common linear phenomena, formulate and graph corresponding linear functions, and interpret the graph's slope and intercepts as properties of the original situation; and
- work fluently with common linear functions that represent the relationships between:
  - the diameter and circumference of a circle;
  - the height and volume of a container with uniform cross-section;
  - the sum of a polygon's interior angles and the number of sides;
  - distance and time under constant speed;
  - measurement units in different systems, such as pounds and kilograms; and
  - total cost of a purchase, cost per unit, and the number of units purchased.

For nonlinear functions, students should understand:

- quadratic functions and their graphs by expressing them in various forms; and
- the concept of a nonlinear function through examples of:
  - simple rational functions, such as $f(x) = \frac{1}{x-1}$;

  - quadratic functions, such as $f(x) = 2x^2 + 4x + 1$;
  - simple cubic functions, such as $V(r) = \quad r^3$;
  - simple square root functions, such as $f(x) = 5\sqrt{x}$; and
  - exponential functions with an integer base, such as $f(n) = 2^n$.

Students should be able to:

- recognize simple nonlinear functions that arise in problem contexts (e.g, quadratic, cubic, rational, exponential) and represent them using tables, graphs and formulas;
- transform and graph quadratic functions by factoring and completing the square; and
- work fluently with common nonlinear functions that represent relationships between:
  - the area and radius of a circle;
  - the volume and radius of a sphere;
  - the number of diagonals and the number of sides of a polygon;
  - the areas of simple plane figures and their linear dimensions;
  - the surface areas and volumes of simple three-dimensional solids and their linear dimensions; and
  - the value of a bank deposit, the interest rate, compounding period and time elapsed.

C. Equations

The concept of an equation and its solutions is a central idea in algebra and opens the door to further study of mathematics, science, business and many other subjects. Equations are related to but different from functions, and it is important that students in the middle grades gain enough experience with both to recognize their similarities and differences. Students also need to learn how to translate problem contexts (presented in words, symbols or "real life" situations) into equations, to solve the resulting equations and then interpret the solution thoughtfully in terms of the original problem context.

Students should understand:

- that to solve the equation $f(x) = g(x)$ means to find all values of x for which the equation is true;
- that the solution to a linear equation corresponds to the point at which the graph of the associated linear function crosses the x axis;
- why the solution of two simultaneous equations is given by the coordinates of the intersection of the graphs of the two corresponding functions; and
- why identical manipulations carried out on both sides of an equation create new equations that have the same solutions as the original (as well as possibly some additional solutions).

Students should be able to:

- use properties of the real number system (e.g., distributive and associative laws) to simplify expressions and solve equations;
- keep equations "balanced" by carrying out the same operations on each side;
- generate and solve linear equations of the form $ax + b = c$ and $ax + b = cx + d$;
- generate linear equations that represent simple problem contexts, manipulate and solve these equations, and interpret these equations in terms of the original context;
- set up and solve simultaneous linear equations in two variables;
- recognize and express correctly the connections between equations and functions;
- use graphs to estimate solutions and check algebraic approaches when solving linear equations;
- relate the solutions of a quadratic equation to the graph of the corresponding quadratic function; and
- solve a quadratic equation by factoring, finding where the graph of the function meets the x axis and completing the square.

Sample Problems: I. Number

Contents

Summary of expectations:

A. Whole numbers:
- Understand and be able to compute factors and multiples
- Find prime numbers and prime factors
- Recognize that the prime factorization of a number is unique

B. Rational numbers:
- Understand and perform arithmetic on rational numbers
- Be able to solve routine and nonroutine problems involving rational numbers
- Understand the ordering of rational numbers on the number line
- Understand the properties of fractions and their decimal representations

C. Real numbers:
- Understand ratio as a comparison of two quantities by division
- Work effectively with ratios, percentages, changes of scale and rates of change
- Recognize that irrational numbers are numbers that cannot be represented as a quotient of integers
- Know some common examples of irrational numbers
- Understand the concept and notation of square and cube roots
- Use integer exponents to express numbers in scientific notation

N1: Square Game

In a party, the following game is played: Each player is given a rectangular piece of graph paper that is 56 centimeters long and 84 centimeters wide. The horizontal and vertical lines are spaced one centimeter apart.

The paper is to be cut along the grid (graph) lines into square pieces that are all the same size without having any paper left over. The winner is the one who cuts the largest square pieces of paper. What would be the length in centimeters of the side of each winning square?

A SOLUTION

The problem is not simply to cut the paper into congruent square pieces with no paper left over, but to cut the paper into the largest possible congruent square pieces with no paper left over. This can be accomplished by finding the largest number that divides evenly into the length 84 centimeters and the width 56 centimeters.

Since 56 and 84 are both even, we could, for example, cut the paper into squares with length of sides at 2 centimeters. We would get 28 rows each with 42 squares.

But both 56 and 84 are also divisible by 4. Cutting the paper into squares that are 4 centimeters on a side would give 14 rows each with 21 squares.

We need to find the largest number that divides both 56 and 84 evenly.

The numbers that divide 56 evenly are the *factors* for 56. The factors can be found by trying the numbers 1, 2, 3, 4, 5 and so on, until the pairs of factors start to repeat.

For example: Try 1. 1 x 56 = 56, so 1 and 56 are factors.

Try 2. 2 x 28 = 56, so 2 and 28 are factors.

Try 3. 3 does not divide 56 evenly, so 3 is not a factor.

Continuing in this way, we find that 4 and 14 are factors, and 7 and 8 are factors. After we try 7, the factors begin to repeat. Therefore, the factors of 56 are 1, 2, 4, 7, 8, 28 and 56. The factors of 84 can be found similarly: $1 \times 84 = 84$; $2 \times 42 = 84$; $3 \times 28 = 84$; $4 \times 21 = 84$; 5 does not go into 84 evenly; $6 \times 14 = 84$; and $7 \times 12 = 84$. Therefore, the factors of 84 are 1, 2, 3, 4, 6, 7, 12, 21, 28, 42 and 84.

The numbers that divide both 56 and 84 are *common factors* of 56 and 84. The greatest number that divides both is the *greatest common factor* of 56 and 84. From the lists above, we see that the common factors of 56 and 84 are 1, 2, 4 and 28. Thus, the greatest common factor is 28.

Since 28 is the greatest number that divides both 56 and 84, the largest squares that can be cut from the rectangle with no paper left over are squares with sides of 28 centimeters.

The 56-centimeter side of the paper would have $\frac{56}{28} = 2$ squares along its edge.

The 84-centimeter side would have $\frac{84}{28} = 3$ squares along its edge.

So a total of 2×3, or 6, squares can be cut from the paper.

The greatest common factor of 56 and 84 can also be found using prime factorization: $56 = 2^3 \times 7$, and $84 = 2^2 \times 3 \times 7$. The greatest common factor contains the common prime factors of the two numbers: $2^2 \times 7 = 28$.

ANOTHER SOLUTION

Let's look at the 56-centimeter side and the 84-centimeter side of the paper separately. Inspecting these tables shows that the largest square that works in both cases is 28 cm on a side.

The 56-cm side could be divided into this many equal pieces:	2	3	4	5	6	7	8	...	...
Doing this would require squares of this size in cm.	**28**	$18\frac{2}{3}$	14	11.2	$9\frac{1}{3}$	8	7	...	...

The 84-cm side could be divided into this many equal pieces:	2	3	4	5	6	7	8	...	...
Doing this would require squares of this size in cm.	42	**28**	21	16.8	14	12	10.5	...	...

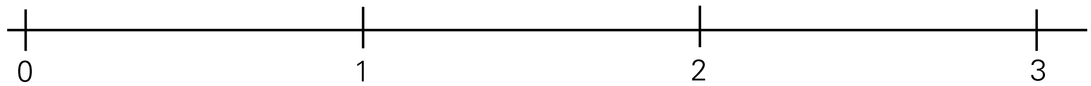
The 56-centimeter side of the paper would have $\frac{56}{28} = 2$ squares along its edge.

The 84-centimeter side of the paper would have $\frac{84}{28} = 3$ squares along its edge.

So there would be a total of 2 x 3 = 6 squares.

Comment: 28 is called a *common factor* of 56 and 84, since it is a factor of both 56 and 84. It is also the greatest number that is a common factor of 56 and 84.

N2: In Between



a) Find the number on the number line that is exactly one third of the way from 1 to 3 (and closer to 1).

b) Find the number on the number line that is exactly halfway between $\frac{2}{9}$ and $\frac{3}{11}$. Is this number more or less than $\frac{1}{4}$?

A SOLUTION

a) To find the number that is exactly one-third of the way between 1 and 3 and closer to 1, we start by dividing the length of the line segment between 1 and 3 into three equal parts. The length of line segment between 1 and 3 is 2 units. When a 2-unit segment is divided into three equal parts, each part is $\frac{2}{3}$ units long. Therefore, in order to go one-third of the way from 1 to 3, we move a distance of $\frac{2}{3}$ units from 1 toward 3, thereby arriving at the point $1 + \frac{2}{3} = 1\frac{2}{3}$.

b) We can use this same method to find the number on the number line that is exactly halfway between $\frac{2}{9}$ and $\frac{3}{11}$. We start with the length of the line segment from $\frac{2}{9}$ to $\frac{3}{11}$, which is $\frac{3}{11} - \frac{2}{9} = \frac{5}{99}$ units.

The length of half of a segment that is $\frac{5}{99}$ units long is $\frac{1}{2} \times \frac{5}{99} = \frac{5}{198}$ units. Thus, to arrive at the halfway point between $\frac{2}{9}$ and $\frac{3}{11}$, we go a distance of $\frac{5}{198}$ beyond $\frac{2}{9}$, which is the point $\frac{2}{9} + \frac{5}{198} = \frac{44}{198} + \frac{5}{198} = \frac{49}{198}$.

Since $\frac{1}{4} = \frac{49.5}{198}$, and $49.5 > 49$, we can conclude that $\frac{1}{4} > \frac{49}{198}$ (or equivalently, that $\frac{49}{198} < \frac{1}{4}$). Alternatively, we can reason as follows: $\frac{1}{4} = \frac{49}{196}$, which has a smaller denominator than $\frac{49}{198}$. Thus, the value of the fraction $\frac{49}{196}$, namely $\frac{1}{4}$, is greater than $\frac{49}{198}$.

ANOTHER SOLUTION FOR PART B

For any two numbers A and B, the number halfway between them is the average of A and B, namely $\frac{(A + B)}{2}$. Thus the number exactly halfway between $\frac{2}{9}$ and $\frac{3}{11}$ is:

$$\frac{\frac{2}{9} + \frac{3}{11}}{2} = \frac{\frac{22}{99} + \frac{27}{99}}{2} = \frac{49}{198}.$$

Since $\frac{1}{4} = \frac{49}{196}$, this number is slightly smaller than $\frac{1}{4}$.

In general, the distance between A and $\frac{(A + B)}{2}$ equals the distance between $\frac{(A + B)}{2}$ and B. If A $\leq$ B:

$\frac{A + B}{2} - A = \frac{(A + B) - 2A}{2} = \frac{B - A}{2}$, and

$B - \frac{A + B}{2} = \frac{2B - (A + B)}{2} = \frac{B - A}{2}$.

A similar calcuation works if A$\geq$B.

N3: Stack of Paper

A stack of 500 identical sheets of paper is 1.9 inches thick. About how many sheets would be needed to make a stack 1 inch thick?

A SOLUTION

If a thickness of 1.9 inches gives 500 sheets, then 1 inch would give $\frac{500}{1.9}$ sheets.

Since $\frac{500}{1.9} \approx 263.15$, 263 sheets would give a stack that is just shy of 1 inch.

ANOTHER SOLUTION

A slightly different approach to this solution is to calculate the thickness of one sheet of paper as follows:

Thickness of one sheet $= \frac{1.9}{500} \approx 0.0038$.

Now divide 1 inch by the thickness of one sheet to find out the number of sheets in a 1-inch stack as follows:

$\frac{1}{0.0038} \approx$ 263.15 sheets.

Again, 263 sheets would give a stack that is just shy of an inch.

N4: Dividing Rational Numbers

a) Without using a calculator, find $\frac{2}{3}$ divided by $\frac{1}{5}$.

b) Using the relationship between multiplication and division, write a multiplication statement that proves your answer is correct.

c) Invent a word problem for which your calculation in the solution for part (a) would provide the answer.

A SOLUTION

a) Use the "invert and multiply" algorithm for dividing rational numbers:

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}, \text{ where } a \neq 0,\ c \neq 0, \text{ and } d \neq 0.$$

So, $\frac{\frac{2}{3}}{\frac{1}{5}} = \frac{2}{3} \times \frac{5}{1} = \frac{10}{3}$.

b) The relationship between multiplication and division states that $\frac{a}{b} = c$ $(b \neq 0)$ is the same as $a = bc$.

So, $\frac{\frac{2}{3}}{\frac{1}{5}} = \frac{10}{3}$ is the same as $\frac{2}{3} \times \frac{5}{1} = \frac{10}{3}$.

Computing $\frac{1}{5} \times \frac{10}{3} = \frac{10}{15} = \frac{2}{3}$ shows that the computation is correct.

c) A word problem that could be solved by the calculation in part (b) is as follows: It takes $\frac{1}{5}$ pound of sugar to fill each restaurant sugar bowl. If you have $\frac{2}{3}$ of a pound of sugar, how many sugar bowls can you fill?

N5: Free Throws

Juan made 13 out of 18 free throws. If Bonita shoots 25 free throws, what's the minimum number she has to make in order to have a better free-throw percentage than Juan?

A SOLUTION

If Bonita makes S shots out of 25 free throws, her shooting percentage would be $\frac{S}{25}$. We want $\frac{S}{25} > \frac{13}{18}$, the latter being Juan's success ratio. If we multiply both sides of an inequality by a positive number, the direction of the inequality is preserved. So let's multiply by 25 to clear the denominator:

$$\frac{S}{25} > \frac{13}{18}$$

$$\frac{S}{25} \times 25 > \frac{13}{18} \times 25$$

$$S > \frac{13 \times 25}{18}$$

$$S > 18\,\frac{1}{18}.$$

Therefore, Bonita needs at least 19 successful free throws (18 is not quite enough) to have a better free-throw percentage than Juan.

ANOTHER SOLUTION

$\frac{13}{18} \approx 0.722$, so Juan made about 72% of his free throws.

Let S = the number of successful free throws Bonita needs to make to have the same percentage as Juan. Then,

$$\frac{S}{25} = 0.722$$

$$\frac{S}{25} \times 25 = 0.722 \times 25$$

$$S = 18.05.$$

Thus, Bonita needs 18.05 successful free throws to have the same percentage as Juan, so she needs more than this to have a better percentage. Therefore, she needs to make 19 successful free throws.

N6: Sale Price

A store is having a sale. Items costing \$30 or more are discounted by 15%, and items costing less than \$30 are discounted by 10%. Suppose Juanita buys four items costing over \$30, which together cost \$140, and five items for under \$30 each, which together cost \$120. Assuming a sales tax of 5%, how much will Juanita have to pay?

A SOLUTION

Juanita bought \$140 worth of items that are discounted by 15%. So the price after the discount is taken is \$119. (15% of \$140 is \$21, and \$140 − \$21 = \$119.)

She also bought \$120 worth of items that are discounted by 10%. So the price after the discount is taken is \$108. (10% of \$120 is \$12 and \$120 − \$12 = \$108.)

The total price for the items will be \$227 (\$119 + \$108).

But there is a 5% sales tax on \$227, which comes to \$227 x 0.05 = \$11.35. Therefore, the amount she pays, including tax, is \$227 + \$11.35 = \$238.35.

The amounts can also be found directly. Since 100% − 15% = 85%, the cost of an item discounted by 15% is 85% of the original price: \$140 x 0.85 = \$119. Similarly, the cost of an item discounted by 10% is 90% of the original price: \$120 x 0.9 = \$108.

Since the sales tax is 5%, the amount paid including sales tax is 100% + 5%, or 105%, of the purchase price. \$227 x 1.05 = \$238.35.

N7: Still Rational

Show that the product of any two nonzero rational numbers is also rational.

A SOLUTION

Pick any two nonzero rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, where a, b, c and d are integers, and $b \neq 0$ and $d \neq 0$.

The product of these two rational numbers is $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

The product of any two integers is an integer. So this means that the products ac and bd are both integers.

So $\frac{ac}{bd}$ is the quotient of two integers. Therefore $\frac{ac}{bd}$ must be rational.

N8: Which Is Larger?

Without using a calculator, decide which is larger: the cube root of 63 or the square root of 18.

A SOLUTION

Since $64 = 4 \times 4 \times 4$, $\sqrt[3]{64} = 4$. However, $63 < 64$. Therefore, $\sqrt[3]{63} < 4$.

Now, $16 = 4 \times 4$, so $\sqrt{16} = 4$. Since $18 > 16$, $\sqrt{18} > 4$.

Thus, $\sqrt[3]{63} < 4$, while $4 < \sqrt{18}$. Hence, $\sqrt[3]{64} < \sqrt{18}$.

N9: On the Number Line

Without using a calculator, mark the approximate location of the following numbers on a number line:
a. $\sqrt{3}$, b. , c. $\sqrt[4]{20}$, d. $\frac{8}{3}$, e. -2, f. 2.5×10^{-3}.

A SOLUTION

a. $\sqrt{4} = 2$, therefore, $\sqrt{3} < 2$.

b. is a little greater than 3 (≈ 3.14).

c. $\sqrt[4]{16} = 2$, therefore, $\sqrt[4]{20} > 2$.

$\sqrt[4]{81} = 3$, so $\sqrt[4]{20} < 3$ (but it appears to be closer to 2 than to 3).

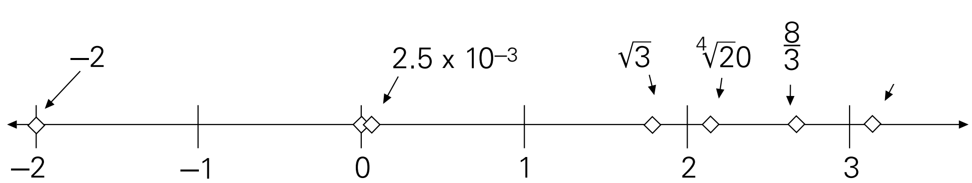
One can confirm this hypothesis by testing 2.5, the midpoint between 2 and 3. Since $25^2 = 625$, $2.5^2 = 6.25$. Thus $(2.5)^4 = 6.25^2 > 36$. So $2.5^4 > 20$ and, thus, the fourth root of 20 is greater than 2 but less than 2.5.

d. $\frac{8}{3} \approx 2.66$. Therefore, $\frac{8}{3}$ is closer to 3 than it is to 2.

e. -2 is less than zero.

f. $2.5 \times 10^{-3} = 0.0025$ is a little greater than 0.

On the number line, these numbers would line up in the following order:



N10: Magic Square

8	3	4
1	5	9
6	7	2

Magic Total = 15

The square above is a *magic square.* Each row, column and diagonal adds up to the same number. This number is called the *magic total.*

a) Complete this magic square:

3		
15		1

Magic Total = 24

b) The magic square below should have a magic total of 36. The number in the center square is represented by x. Express each of the other numbers in the square in terms of x.

15		
$13-x$	x	23
	7	

Magic Total = 36

c) Write an equation that can be used to find x, and solve it.

A SOLUTION

a) Since the middle row must add to 24, the number in the middle has to be 8. The sum along the diagonal containing 3 and 8 again must add up to 24, so the lower right corner has to be 13. Similarly, we determine the remaining entries.

3	11	10
15	8	1
6	5	13

Magic Total = 24

b) To find the missing expressions, consider the rows and columns for which two entries are given.

Lower left box:

$$15 + (13 - x) + ? = 36$$

$$? = 36 - 15 - 13 + x$$

$$? = 8 + x.$$

Upper middle box: $7 + x + ? = 36$, so $? = 29 - x$.

Lower right box: $15 + x + ? = 36$, so $? = 21 - x$.

Upper right box: $(8 + x) + x + ? = 36$, so $? = 28 - 2x$.

Using these results to fill in the empty boxes, we get:

15	$29-x$	$28-2x$
$13-x$	x	23
$8+x$	7	$21-x$

Magic Total = 36

c) Let's take one row (the top one, in this example) and see if we can make an equation:

$15 + (29 - x) + (28 - 2x) = 36.$

To solve, we combine terms and simplify:

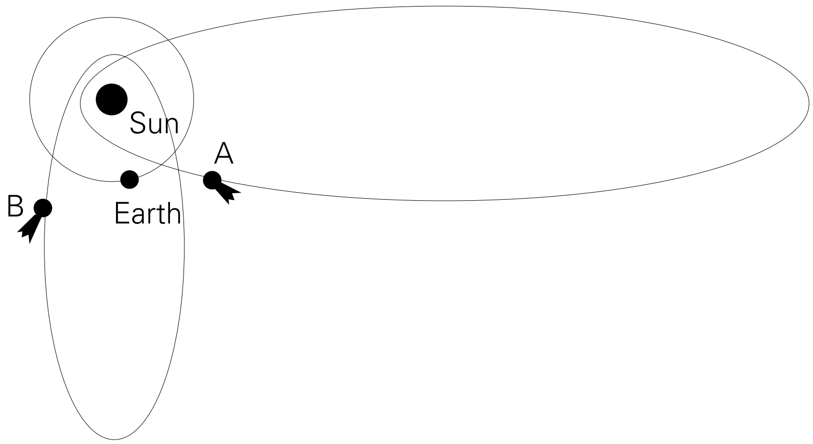
$$\begin{aligned} 72 - 3x &= 36 \\ 3x &= 36 \\ x &= 12. \end{aligned}$$

If we substitute the value $x = 12$ into the magic square, we get:

15	17	4
1	12	23
20	7	9

Magic Total = 36

N11: Two Comets



There are two comets, A and B. A comes close enough to the earth to be observed every 76 years. B comes close enough to the earth to be observed every 12 years.

If we can observe both of the comets from the earth this year, how many years will it be before we can again see them both in the same year?

A SOLUTION

If both comets are visible this year:

Comet A will be visible again in 76 years, in 2 x 76 = 152 years, in 3 x 76 = 228 years, in 4 x 76 = 304 years, and so on; 76, 152, 228, 304 … are *multiples* of 76.

Comet B will be visible again in 12 years, in 2 x 12 = 24 years, in 3 x 12 = 36 years, in 4 x 12 = 48 years, and so on; 12, 24, 36, 48 … are *multiples* of 12.

The comets will be visible at the same time in the first year that is both a multiple of 12 and a multiple of 76. Multiples of both 12 and 76 are *common multiples* of 12 and 76. The first common multiple of the two numbers is the *least common multiple* of 12 and 76.

One way to find the least common multiple of 12 and 76 is to list multiples of each number until we find the first common multiple.

Multiples of 76: 76, 152, **228**, 304, 380 …

Multiples of 12: 12, 24, 36, 48, 60, 72, 84, 96, 108, 120, 132, 144, 156, 168, 180, 192, 204, 216, **228** …

228 is the least common multiple of 12 and 76, so the next time both comets will appear together is in 228 years. Future appearances after that will occur in years that are multiples of 228 (i.e., in $2 \times 228 = 456$ years, in $3 \times 228 = 684$ years, and so on).

Another way to find the least common multiple of 12 and 76 is to list the multiples of the greater number, 76, and test each to see if it is divisible by 12 (if it is divisible by 12, it is also a multiple of 12). The first multiple of 76 that is divisible by 12 is 228.

The least common multiple of 12 and 76 can also be found using prime factorization: $12 = 2^2 \times 3$ and $76 = 2^2 \times 19$. The least common multiple must contain each prime factor the greatest number of times it occurs in each number, so the least common multiple is $2^2 \times 3 \times 19 = 228$.

ANOTHER SOLUTION

If we divide the period of A by the period of B, we get $76 \div 12 = 6\frac{1}{3}$.

This means that:

- when A makes its first return, B has completed $6\frac{1}{3}$ revolutions.
- when A makes its second return, B has completed $12\frac{2}{3}$ revolutions.
- when A makes its third return, B has completed $18\frac{3}{3} = 19$ revolutions.

This is the first time that a return of A coincides with a whole number of revolutions of B. But that is just what is required for A and B to be seen together at the earth. This happens after $3 \times 76 = 19 \times 12 = 228$ years.

Sample Problems: II. Data

Contents

Summary of expectations:

A. Measurement and approximation:
- Understand the metric and English measurement systems
- Convert between units within each system
- Use derived units (e.g., miles per hour)
- Understand approximation, precision and accuracy
- Select and interpret measurements appropriate to specific contexts

B. Data analysis:
- Represent and interpret single-variable data using a variety of methods
- Use relative and cumulative frequencies
- Understand, calculate and reason with median, mean, ranges and quartiles
- Use scatter plots to represent and interpret two-variable data

C. Probability:
- Solve simple problems involving probabilities and relative frequencies
- Compute probabilities of possible outcomes from tossing dice, flipping coins and other simple experiments
- Recognize why inferences about large populations drawn from data samples are never certain

D1. Buying Alaska

In 1867, the United States purchased the 586,412 square miles of Alaska from Russia for \$7.2 million.

a) There are 640 acres in one square mile. About how much did Alaska cost per acre?

b) There are 5,280 feet in one mile. About how much did Alaska cost per square foot?

A SOLUTION

a) There are 586,412 square miles in Alaska, and there are 640 acres in each square mile. Multiplying these two numbers gives us 375,303,680, which is the total number of acres in Alaska. If 375,303,680 acres cost \$7,200,000, then 1 acre cost $\frac{7,200,000}{375,303,680}$ dollars.

$\frac{7,200,000}{375,303,680} \approx \frac{7}{350} = \frac{1}{50}$, or 0.02. Therefore, it cost about 2 cents per acre to purchase Alaska.

b) Multiplying the 586,412 square miles that make up Alaska by $5,280^2$ gives us the size of Alaska in square feet: $586,412 \times (5,280)^2 \approx 1.635 \times 10^{13}$ square feet.

If 1.635×10^{13} square feet cost \$7,200,000, then one square foot cost

$$\frac{\$7,200,000}{1.635 \times 10^{13}} = \frac{\$7.2 \times 10^6}{1.635 \times 10^{13}} \approx \$4.4 \times 10^{-7}.$$

Expressed in decimal notation, this is 0.00000044 dollars, or 0.000044 cents per square foot.

D2. How Big Is the Cube?

Suppose we measure the length, width and height of a cube, and we get 2 inches on each side. We conclude that the volume of the cube is 8 cubic inches. However, we know that the instrument we used to measure the lengths has an error of at most 3%.

a) Between what values could the lengths of the sides actually be?

b) Between what values could the volume of the cube actually be?

c) If the error was the worst possible — a full 3% — what will be the percentage error in measuring the volume of the cube?

A SOLUTION

a) 3% of 2 inches is 0.06 inches. If there is an error of at most 3% when measuring a 2-inch edge of the cube, then the largest possible error is 0.06 inches. From this, the simple conclusion is that the true size of an edge of the cube is between 1.94 inches and 2.06 inches.

However, the true length of an edge of the cube may be more or less than 2 inches (the 2 inches is what we measured, not the true length of the side), so the 3% should really be ±3% of the *true size*, not of the measurement. So, to find the maximum and minimum true size, let L be the true length and solve these equations:

$$1.03\, L_{min} = 2 \qquad L_{min} = \frac{2}{1.03} \approx 1.942 \text{ inches}$$

and

$$0.97\, L_{max} = 2 \qquad L_{max} = \frac{2}{0.97} \approx 2.062 \text{ inches.}$$

So the true length of the cube edge could be between 1.942 inches and 2.062 inches.

b) Using the smallest true length for the edge of the cube, 1.942 inches, the smallest possible volume would be $(1.942)^3 \approx 7.324$ in^3. Using the largest true length for the edge of the cube, 2.062 inches, the largest possible volume would be $(2.062)^3 \approx 8.767$ in^3.

Therefore, the volume could be between 7.324 in^3 and 8.767 in^3.

c) In calculating the largest possible percentage error in measuring the volume of the cube, we need to write a ratio to compare the largest and smallest possible volumes with the measured volume of 8 in^3.

$\frac{7.324}{8} \approx 91.6\%$, so 7.324 in^3 is about 8.4% smaller than 8 in^3.

$\frac{8.767}{8} \approx 109.6\%$, so 8.767 in^3 is about 9.6% bigger than 8 in^3.

Thus, the percentage error in measuring the volume of the cube would be about 9% in either direction.

D3. Airplane Speed

In Canada, they measure distances in kilometers. One kilometer is about 60% of one mile.

a) An airplane going 300 miles per hour is going *S* kilometers per hour. Using the information above, estimate *S*.

b) Estimate this same speed measured in both meters per hour and meters per second.

c) On graph paper, draw a graph that shows the relationship between miles per hour and meters per second using the conversion factor 1 km = 0.6 mi. Label the axes and indicate the scales clearly.

d) From the graph, estimate how many meters per second correspond to 125 miles per hour.

e) This graph represents a proportional relationship between miles per hour and meters per second. What is the constant of proportionality? (This is the slope of the line defined by the given data points and the origin.)

A SOLUTION

a) 300 is 60% of 500, so 300 miles per hour is about the same speed as 500 kilometers per hour.

b) In order to convert 500 km per hour into meters per hour, we need to know that there are 1,000 meters in each kilometer. So we multiply by 1,000 to convert. Notice how we can make the dimensions cancel:

$$\textit{speed} = 500\,\frac{\text{km}}{\text{h}} \times 1000\,\frac{\text{mi}}{\text{km}} = 500{,}000\,\frac{\text{mi}}{\text{h}}.$$

To convert 500 km per hour into meters per second, we begin with the previous result: 500 km per hour is the same speed as 500,000 meters per hour. Dividing by 60 (because there are 60 minutes for every hour) gives us the speed in meters per minute. Notice how, in the calculation, we're multiplying by 1.

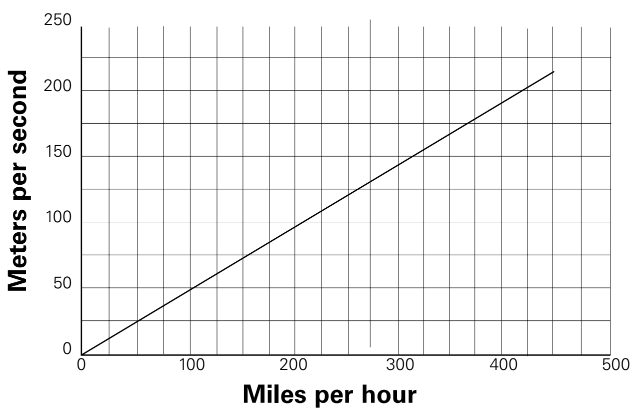
$$\textit{speed} = 500{,}000\,\frac{\text{mi}}{\text{h}} \times \frac{1\text{ h}}{60\text{ min}} = \frac{500{,}000\text{ m}}{60\text{ min}}.$$

Dividing once again by 60 (because there are 60 seconds in a minute) gives us the speed in meters per second.

$$speed = \frac{500{,}000 \text{ m}}{60 \text{ min}} \times \frac{1 \text{ min}}{60 \text{ sec}} = \frac{500{,}000 \text{ m}}{3{,}600 \text{ sec}} \approx 139 \frac{\text{m}}{\text{sec}}.$$

c) To draw the graph that shows the relationship between *miles per hour* and *meters per second*, we need to see that the relationship is linear, that zero miles per hour corresponds to the point (0, 0) on this graph, and that 300 miles per hour is about 139 meters per second, giving us the point (300, 139) on the graph. We can connect the origin through this point to make our graph (below).

MILES PER HOUR VS. METERS PER SECOND



d) Finding 125 miles per hour on the horizontal axis and following the grid line up, we can see that 125 miles per hour is nearly halfway between 50 and 75 meters per second. 62 would be a good estimate.

e) We see from the graph and from the calculations in problems part (a) and (b) that 300 miles per hour on the horizontal axis corresponds to 139 meters per second on the vertical axis. So the equation for the line is $k = \frac{139}{300}m$, where k is speed in meters per second and m is speed in miles per hour. (This way, when m is 300, k is 139, which is what we want.)

The slope of the line, which is the constant of proportionality, is $\frac{139}{300} \approx 0.46$.

D4. Units of Temperature

Temperatures are usually measured in degrees Celsius (C) or degrees Fahrenheit (F). The relationship between C and F is linear. When the temperature is 0º C, it is 32º F. When the temperature is 100º C, it is 212º F.

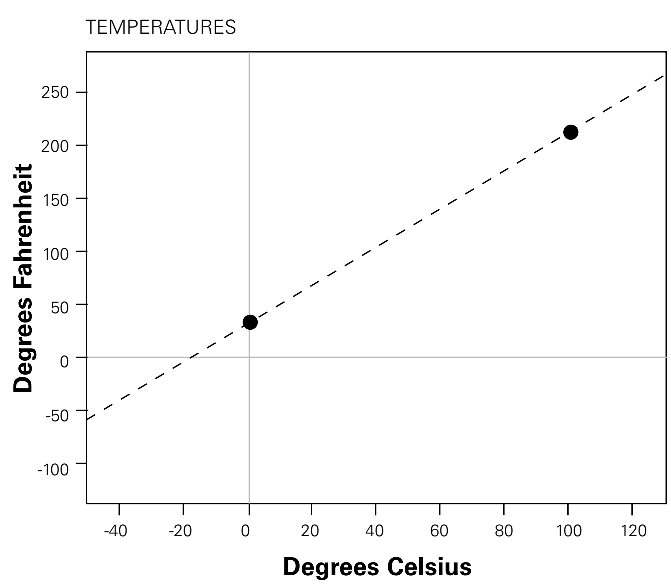
a) Make a graph that shows the relationship between F and C. Don't forget that there are negative temperatures, too.

b) When we say, "It is below zero outside," we usually mean, "below 0° F." What range of Celsius temperatures correspond to "below zero Fahrenheit?"

c) If we make a mistake in measurement by 9º F, how much of a mistake is this in degrees C?

A SOLUTION

a) Since the relationship between degrees Celsius (C) and degrees Fahrenheit (F) is linear, we need only find the two different points that lie on the graph. When the temperature is 0 degrees Celsius (the freezing point of water) it is 32 degrees Fahrenheit, so (0, 32) is one point on the graph. When the temperature is 100 degrees Celsius (the boiling point of water) it is 212 degrees Fahrenheit. Hence (100, 212) is another point on the graph. The straight line connecting these points appears in the figure below.



b) Looking carefully at the graph shows that Fahrenheit temperatures are below zero when Celsius temperatures are less than about −18 degrees. To find this value exactly it is easiest to first find the equation of the line. The line has slope $\frac{212-32}{100-0} = \frac{180}{100} = \frac{9}{5}$.

Since the line goes through the point (0,32), its equation (in point-slope form) is
$F - 32 = \frac{9}{5}(C - 0)$ or $F = \frac{9}{5}C + 32$.

Solving for C gives the equation: $C = \frac{5}{9}(F - 32)$.

From this last equation, setting $F = 0$ gives us $C = -\frac{5}{9}(32) = -\frac{160}{9}$, or about −17.7 degrees Celsius.
F is less than 0 if and only if C is less than −17.77.

c) We can see from the graph that every difference of 9° F corresponds to a difference of 5° C. A mistake of 9° F means a rise (or fall) of 9° on the vertical scale of the graph, which corresponds to a rise (or fall) of 5° on the horizontal scale because the slope of the graph is $\frac{9}{5}$. So a 9° error in F produces a 5° error in C.

D5. Crop Yields

A tomato farmer can get five pounds of tomatoes from each square yard planted in her vegetable garden. At that rate, how many tons of tomatoes would she get per acre? (There are 2,000 pounds in a ton, and 4,840 square yards in an acre.)

A SOLUTION

Start with the given ratio of 5 pounds of tomatoes for every square yard. If we multiply that ratio by 4,840 (because there are 4,840 square yards in an acre), we get the number of pounds of tomatoes per acre:

$$5 \frac{\text{pounds}}{\text{square yard}} \times 4{,}840 \frac{\text{square yards}}{\text{acre}} = 24{,}200 \frac{\text{pounds}}{\text{acre}}.$$

Divide this 24,200 by 2,000 (because there are 2,000 pounds in a ton) to get the number of tons of tomatoes per acre:

$$\frac{24{,}200 \frac{\text{pounds}}{\text{acre}}}{2{,}000 \frac{\text{pounds}}{\text{ton}}} = \frac{24{,}200 \text{ pounds}}{1 \text{ acre}} \times \frac{1 \text{ ton}}{2{,}000 \text{ pounds}}$$

$$= 12.1 \text{ tons per acre.}$$

D6. 50-Meter Dash

Time (sec.)	Frequency
7.0–7.3	1
7.4–7.7	4
7.8–8.1	9
8.2–8.5	12
8.6–8.9	10
9.0–9.3	3
9.4–9.8	1

The table above gives the times in the 50-meter dash for 40 eighth-grade girls, organized into a frequency distribution table.

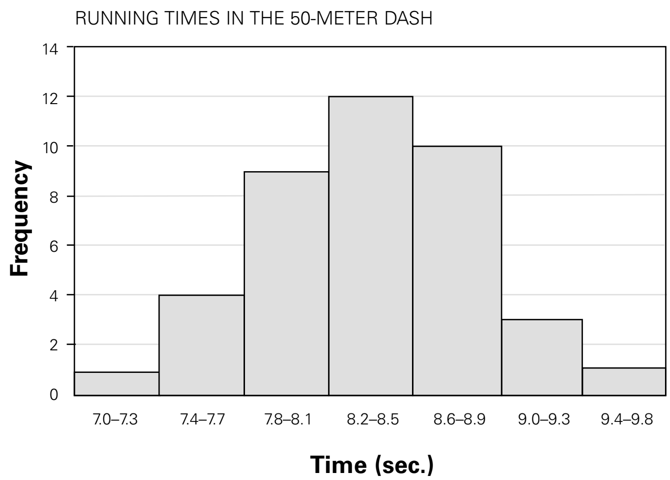
a) Make a histogram that shows the data given in the table.

b) Make a relative frequency distribution table.

c) Make a cumulative frequency distribution table and a broken line graph of the cumulative frequency.

A SOLUTION

a) This histogram shows the number of girls who ran the 50-meter dash in each of seven different time intervals. The intervals are all the same, namely 0.4 seconds.



b) To add the relative frequency column, we use the fact that there are 40 recorded running times. In the relative frequency column, we write each frequency as a percentage of 40; for example one runner out of 40 recorded a running time in the interval 7.0–7.3 seconds. One out of 40 is 2.5%.

RUNNING TIMES IN THE 50-METER DASH		
Time (sec.)	**Frequency**	**Relative Frequency**
7.0–7.3	1	2.5%
7.4–7.7	4	10.0%
7.8–8.1	9	22.5%
8.2–8.5	12	30.0%
8.6–8.9	10	25.0%
9.0–9.3	3	7.5%
9.4–9.8	1	2.5%
Total	40	

c) The cumulative frequency at each time interval is the sum of all the interval frequencies up to and including that interval. For example, the fifth cumulative frequency entry is the sum of the frequencies for the first five intervals.

$1 + 4 + 9 + 12 + 10 = 36.$

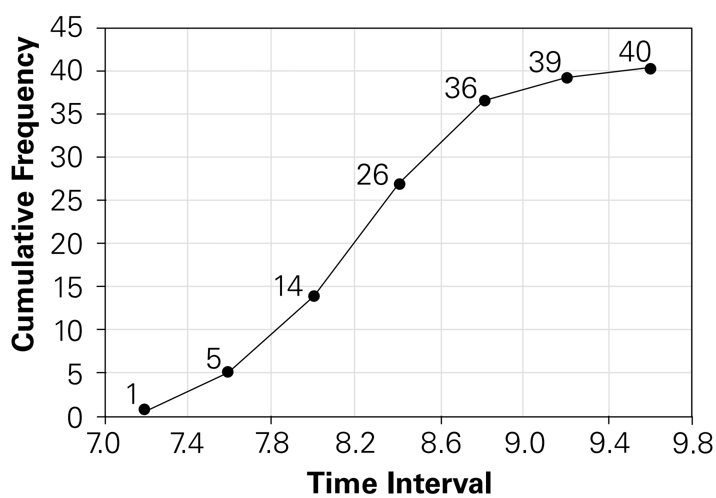
This also is the sum of the fourth cumulative frequency (namely, $1 + 4 + 9 + 12 = 26$) and the frequency of the fifth interval. This is how cumulative frequencies are usually calculated — by adding new frequencies to prior accumulated totals — whence the name "cumulative" frequency.

RUNNING TIMES IN THE 50-METER DASH

Time (sec.)	Frequency	Cumulative Frequency
7.0–7.3	1	1
7.4–7.7	4	5
7.8–8.1	9	14
8.2–8.5	12	26
8.6–8.9	10	36
9.0–9.3	3	39
9.4–9.8	1	40

The broken line graph shows the cumulative frequency for each interval. The broken line ends at the seventh interval because there are only seven intervals.

CUMULATIVE FREQUENCY OF RUNNING TIMES



D7. Home Gardens

In the *Statistical Abstract of the United States* (1981, p. 232), the median size of a home garden is listed as 663 square feet.

a) Explain what it means to say that the median size of a home garden is 663 square feet.

b) Why do you think they reported the median size of a home garden instead of the mean?

c) Is it possible for the upper-quartile home garden size to be less than the mean home garden size?

A SOLUTION

a) It means that half the home gardens are 663 square feet or smaller, and half are 663 square feet or larger.

To find the median size, arrange all the garden sizes in numerical order from least to greatest. Then find the number that is exactly in the middle of this list. In this case, it was 663 square feet. (If there are two numbers in the middle, the median would be the average of these two numbers.)

b) It is likely they reported the median because there were a few unusual gardens in the distribution — that is, extremely large — making the mean rather large and not typical. If one has a list of gardens in more or less the same range of sizes, and then adds one or two "gardens" the size of New York's Central Park, the mean size of the new list of gardens has increased appreciably, while the median is unchanged. However, it is a matter of judgment which measure — median or mean — is more "typical," and the choice depends on the use one wishes to make of the number.

c) Yes, this is possible, as the following example shows. Use these garden sizes:

{100, 110, 120, 130, 150, 170, 180, 190, 200, 600, 800}

The median is 170 (the middle number when the data are placed in numerical order). The upper quartile is the median of the numbers above 170, in this case 200.

The upper quartile is 200, while the mean is:

$$\frac{100 + 110 + 120 + 130 + 150 + 170 + 180 + 190 + 200 + 600 + 800}{11} = 250.$$

The median is 170, even farther below the mean than the upper quartile, but it is not correct to say that 170 is more or less valid than 250 as a measure of a "typical" garden.

Because medians and quartiles are determined only by their position on the list when in numerical order, the magnitude of the largest and smallest values — which influence the mean — have no effect on medians or quartiles.

D8. Mathematics Scores

Suppose all of the students in your school are given the same mathematics exam, and the following data are reported:

Low score:	24
Mean:	42
Lower quartile score:	44
Median:	48
Upper quartile score:	56
High score:	62

a) Give a range of scores so that 50% of the students at your school scored within that range.

b) Give another range different from the one above that contains 50% of the students' scores.

c) Suppose 522 students at your school scored between 24 and 56 on the exam. How many total students took the exam at your school?

d) Suppose your friend got a score of 46 on the exam. What can you say about your friend's performance on the exam relative to the other students at your school?

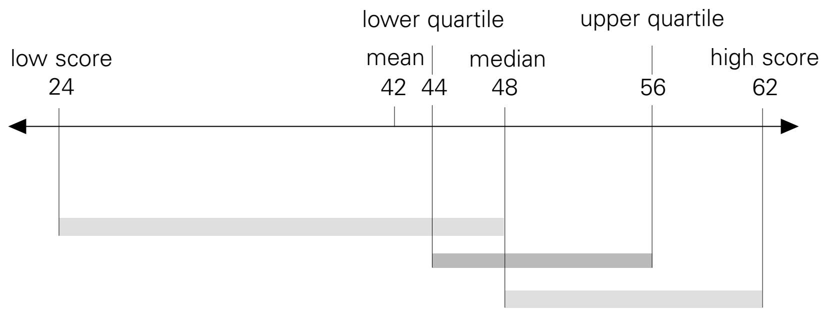
A SOLUTION

a) The two quartiles and the median separate the range of mathematics scores into four intervals, each having an equal number of test scores. Choosing any two adjacent groups would give a range of scores with 50% of all test scores within that range.

Thus, because 44 is the lower quartile and 56 the upper quartile, a range of 44 through 56 would represent 50% of all the test scores.

b) Other ranges that would also represent 50% of the scores would be all the scores at or below the median, or all the scores at or above the median. By definition, the median is the score exactly in the middle of all the test scores when placed in numerical order. Thus, the range 48 through 62, or 24 through 48, would represent 50% of the scores.

The illustration below shows three possible answers to parts (a) and (b) as gray bands.



c) **Simple answer:** The range of 24 to 56 represents all the test scores from the lowest score to the upper quartile. That means this range represents $\frac{3}{4}$ of all the students.

Since 522 students is $\frac{3}{4}$ of all the students, we can find the total number of students by solving the following equation, where x is the total number of students.

$$\frac{3}{4}x = 522$$

$$x = 522 \times \frac{4}{3}$$

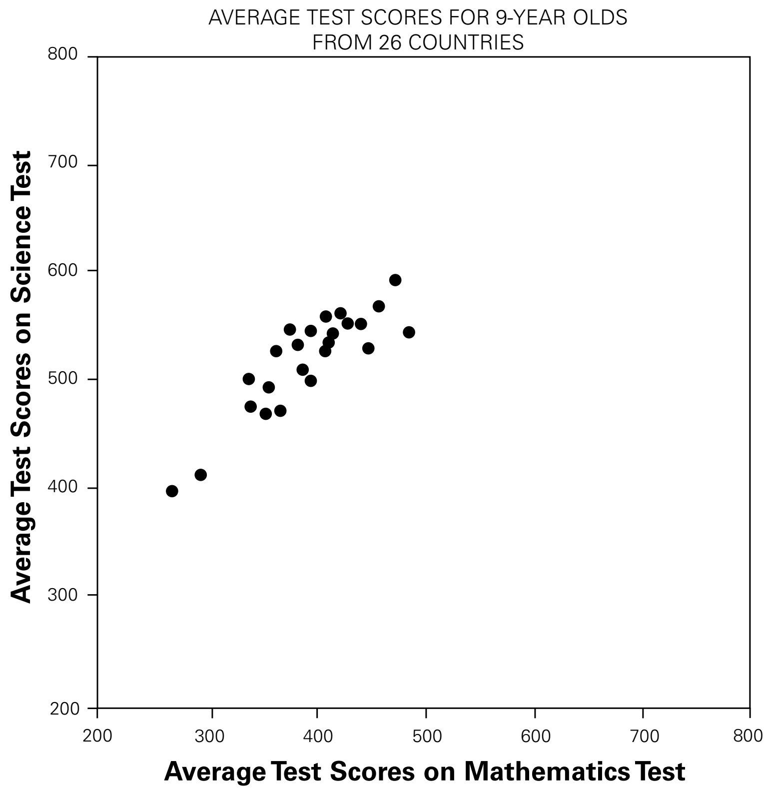
$$x = 696 \text{ students.}$$

Attending to more complicated issues: We cannot exactly calculate the number of students. The above answer, 696, is a possibility. But if we added another student right at the top score, the upper quartile might not change. Then we'd have 697 students and exactly the same situation!

Or, suppose that there are 523 students in the school, but among the top 131 scores, 130 students score 56, and one student scores 62. Then it's possible that the upper quartile is at the "bottom" end of the clump of 56s, and all of the statistics are the same. So there is a range of answers, running from 523 at least to 697.

d) A student scoring 46 has placed a little below the median, or one might say, rather high in the bottom half of the class. Sometimes people might say that for this class 46 is "below average." We must be careful, though. "Average" usually means "mean." In that case, saying 46 was "below average" would be wrong, since the mean is actually 42.

D9. Mathematics and Science Correlation



The scatter plot above shows the average test scores on two different tests — one for mathematics and one for science — for 9-year olds from 26 different countries. (Note that the origin of the graph is at (200, 200) instead of (0, 0), since the scale of these tests runs from 200 to 800).

a) Is there a positive association, a negative association or no association between mathematics scores and science scores?

b) Draw in the line that you feel best represents the data in the scatter plot, and find an equation for the line.

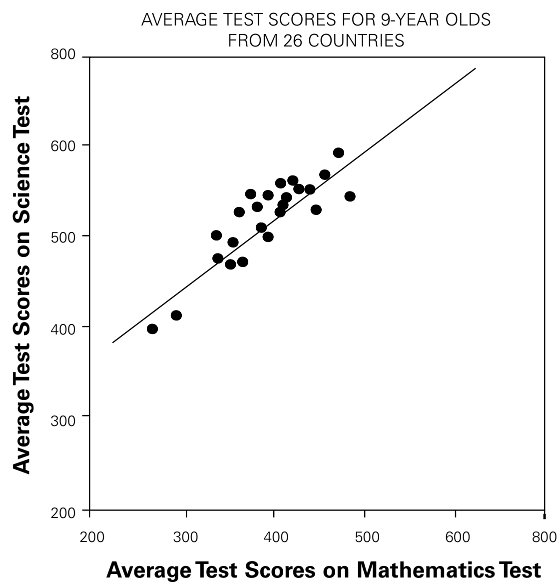
c) Suppose another country's children averaged 460 on their mathematics scores, but the data are not shown in the scatter plot. What would you expect the average science test score to be for children in this country?

d) Suppose someone said, "The scatter plot above shows that being good at mathematics causes you to have high science scores as well." Do you think this is a valid conclusion from the scatter plot?

A SOLUTION

a) There is a positive association between the mathematics and science scores in that, by and large, points showing higher scores for mathematics also show higher scores for science.

b) In "eyeballing" a line of best fit, there is no perfect answer. The graph below shows one possibility. The line drawn on the graph represents the data about as well as a line drawn by a visual estimate can, in that it seems to have roughly as much "error" represented by points of the scatterplot above it as below it.



To find the equation of this line, we begin with the slope-intercept formula, $y = mx + b$, where x represents the mathematics score and y represents the science score. We need to find the values of m and b, where m represents the slope of the line and b represents the y-intercept of the line.

The easiest way to find the slope of this line is to find two points that lie on the line, whether or not those points are part of the scatterplot, and use them to calculate the slope. It appears, for example, that the points (300, 445) and (600, 670) lie on this line.

Using these two points, we can calculate the slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{670 - 445}{600 - 300} = \frac{225}{300} = \frac{3}{4}.$$

Now our equation looks like $y = \frac{3}{4}x + b$.

To find the value of b we can substitute the x and y values of the point (300, 445) into the equation and solve for b:

$$445 = \frac{3}{4}(300) + b = 225 + b$$

$$b = 220.$$

Thus, the equation that represents our "eyeballed" line of best fit is $y = \frac{3}{4}x + 220$.

c) A country with a mathematics score of 460 probably will have a science score of about 565. We make this prediction because 460 corresponds with 565 on the line of best fit.

Another way to make this prediction is to use the formula created in part (b) and substitute 460 for the value of x:

$$y = \frac{3}{4}(460) + 220 = 345 + 220 = 565.$$

d) If someone were to conclude that the scatter plot shows that being good at mathematics causes you to have high science scores as well, this would not be a valid conclusion. While the scatterplot shows a positive association (or correlation) between mathematics scores and science scores, it does not show us that one causes the other.

In other words, being good at mathematics does not necessarily mean that you will be good at science, since there may be other factors contributing to high science scores besides just high mathematics scores. Without more evidence, one cannot conclude that higher mathematics scores are the cause of higher science scores, or vice versa. It might be that they are correlated because they are both related to a common factor, for example, literacy or good schools.

D10. Flipping a Penny

You conduct an experiment by flipping a penny twice to see if you get two heads. You do this a total of 25 times, and seven times you get two heads.

a) Based on your experiment, what appears to be the relative frequency of getting two heads when you flip a coin twice?

b) Assuming that the relative frequencies are equal for heads and tails, (which is not what the experiment in part (a) showed, but which is true in the long run for "fair" coins), and that the result of a second toss is independent of the result of the first, what is the probability of getting two heads when you flip a coin twice?

A SOLUTION

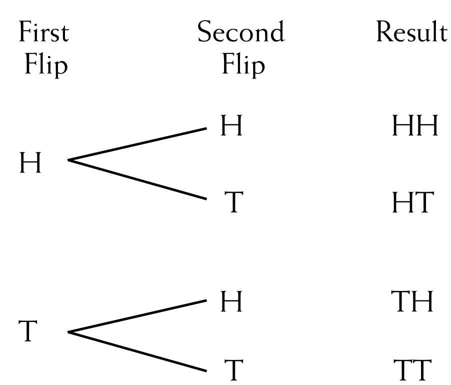
a) Based on our experiment, the fraction of the time that we get two heads is $\frac{7}{25}$ of the time. If we want the relative frequency of getting two heads, we need to change $\frac{7}{25}$ into a percent, which gives a relative frequency of 28% of the time.

b) Let's make a tree diagram to identify all the possible outcomes when flipping a coin twice. In the first column, we write down the only two possibilities for the first flip: The coin can either land on a head or a tail.

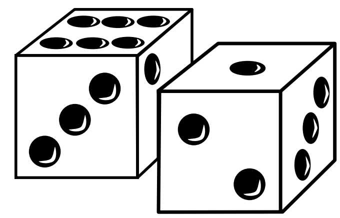
In the second column, we need to record the two possibilities of the second flip, first for the first flip being a head and then again for the first flip being a tail.

The third column shows the results.

As we can see in the result column, there are four possible outcomes, one of which has heads showing up twice. Therefore, the probability of getting two heads when flipping a coin twice is $\frac{1}{4}$, or 25%.



D11. Rolling Dice



You roll a pair of fair six-sided dice.

a) What is the probability that the sum of the numbers on the uppermost faces of the dice will be 6?

b) What is the probability that you will roll doubles?

c) Are "rolling a sum of 6" and "rolling doubles" equally likely events?

A SOLUTION

a) To find the probability of getting a sum of 6 when rolling two dice, we need to first identify the total number of possible outcomes. Since both die are six-sided, the total number of possible outcomes is $6 \times 6 = 36$.

Now we need to identify which of the 36 outcomes have a sum of 6. Listing the outcomes as ordered pairs, they are (1, 5), (2, 4), (3, 3), (4, 2) and (5, 1). That is, there are five combinations with a sum of 6.

When calculating the probability of an event, the numerator is the number of favorable outcomes, and the denominator is the total number of possible outcomes. Thus,

$$P(\text{sum of } 6) = \frac{\text{number of outcomes with a sum of } 6}{\text{total number of outcomes when rolling two dice}} = \frac{5}{36}.$$

b) From part (a) we know that there are 36 possible outcomes.

There are six combinations in which doubles are rolled: (1, 1), (2, 2), (3, 3), (4, 4), (5, 5) and (6, 6). Hence,

$$P(\text{rolling doubles}) = \frac{\text{number of combinations involving doubles}}{\text{total number of outcomes when rolling two dice}} = \frac{6}{36} = \frac{1}{6}.$$

c) "Rolling a sum of 6" and "rolling doubles" are not equally likely events because $\frac{5}{36} \neq \frac{1}{6}$.

D12. Rolling Dice With Conditions

a) Suppose you were rolling a pair of dice and had just rolled the "double" (2, 2) and the "double" (4, 4). On your next roll, what is the probability of rolling another "double"?

b) Suppose you rolled the dice into a place you couldn't see, but your friend looked and reported that the total of the two dice was an even number. What is the probability, given that information, that you had rolled a "double"?

c) Suppose instead that your friend reported that the total of the two dice was an odd number. What is the probability, given this information, that you had rolled a "double"?

A SOLUTION

a) Assuming an honest pair of dice, the probability of "doubles" is the same as it would have been if the preceding two rolls had been anything else. That is, it would be $\frac{6}{36}$, or $\frac{1}{6}$, because there are six possible "doubles" and 36 equally likely possible outcomes of a roll. The future behavior of honest dice is independent of their past behavior.

b) Here the answer is different. Knowing that the outcome is even means that you have rolled one of the following 18 possible even outcomes: (1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (2, 6), (3, 1), (3, 3), (3, 5), (4, 2), (4, 4), (4, 6), (5, 1), (5, 3), (5, 5), (6, 2), (6, 4) and (6, 6). Of these, six are "doubles." So the probability is $\frac{6}{18}$, or $\frac{1}{3}$, that — given an even roll — the roll is a double.

(Recall that in part (a), where all outcomes are regarded as equally likely, the probability of a double was only $\frac{1}{6}$. Advance knowledge that the outcome is even improves the probability of a double.)

c) Every time you roll a double, the sum is even. So the probability that the roll is a double, given that it is odd, is zero.

D13. Ice Cream or Cake?

Suppose you survey all the students at your school to find out whether they like ice cream or cake better as a dessert, and you record your results in the contingency table below:

a) What percentage of students at your school prefers ice cream over cake?

b) At your school, are those preferring ice cream more likely to be boys or girls?

c) At your school, are girls more likely to choose ice cream over cake than boys are?

	ice cream	cake	totals
boys	82	63	145
girls	85	73	158
totals	167	136	303

A SOLUTION

a) We know that there are 303 students at the school and 167 of them prefer ice cream. Since $\frac{167}{303} \approx 55\%$, this means that about 55% of the students prefer ice cream.

b) Those who prefer ice cream are more likely to be girls since, in this survey, there are more girls who like ice cream than boys.

c) To identify the group more likely to choose ice cream over cake, we first find the percentage of girls who prefer ice cream and the percentage of boys who prefer ice cream.

There are 158 girls total, and 85 of them prefer ice cream. Let's calculate a percentage: $\frac{85}{158} \approx$ 54% of the girls prefer ice cream.

There are 145 boys total, and 82 of them prefer ice cream; again, we calculate a percentage: $\frac{82}{145} \approx$ 57% of the boys prefer ice cream.

Therefore, boys (57%) are more likely than girls (54%) to prefer ice cream.

Even though an ice cream lover is more likely to be a girl than a boy, we cannot conclude that the chance of preferring ice cream is greater among girls than among boys. (This sort of false conclusion is common in public discourse.)

Sample Problems: III. Geometry

Contents

Summary of expectations:

A. Geometric figures:

- Understand the concepts of length, area, volume and surface area
- Understand the concept of angle measure and basic properties of angles
- Know the definitions of radius, diameter and circumference of a circle, and the relationships among them
- Be thoroughly familiar with the properties of triangles (isosceles, equilateral, right), rectangles, parallelograms, trapezoids and other convex polygons
- Recognize two classes (cones and cylinders) of common three-dimensional solids
- Be able to represent three-dimensional figures accurately in two-dimensions

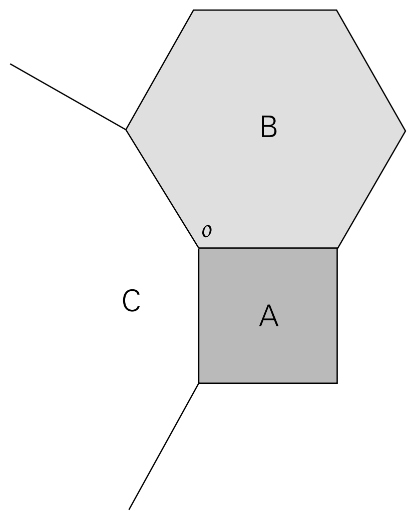
B. Measurement:

- Solve problems involving length, area and volume of common geometric objects
- Understand, prove and use the Pythagorean theorem

C. Transformations:

- Understand the meaning of terms such as rotation, reflection, translation, expansion, contraction and scale factors
- Know how to decide whether two figures are similar or congruent
- Use congruence and similarity in solving problems

G1. Where Polygons Collide



The figure above is part of a figure where three regular polygons, A, B and C come together. A is a square. B is a regular hexagon.

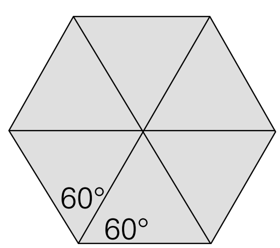
a) What is the size of angle *o*?

b) What is the size of each *exterior* angle of polygon C?

c) How many sides does polygon C have?

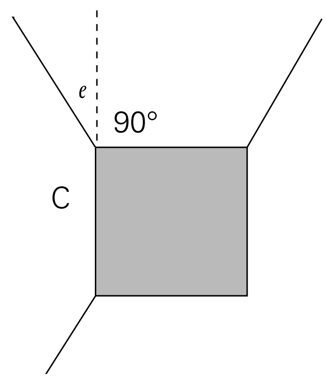
A SOLUTION

a) Angle *o* is an interior angle of a regular hexagon. You can partition this hexagon into six congruent, equilateral triangles that all meet at a point at the center of the hexagon. Angle *o* is made up of two base angles of adjacent triangles. Since each angle of an equilateral triangle is 60°, being $\frac{1}{3}$ of 180°, angle *o* must be $60° + 60° = 120°$.



b) The exterior angle of polygon C is the angle *e*, formed by extending one side. Since angle *o* is 120° and $\angle o = 90° + \angle e$, it follows that $\angle e = 30°$.

Thus, each exterior angle of the regular polygon *c* is 30°.



c) The sum of the exterior angles of a polygon is 360°. C is a regular polygon, where each exterior angle equals 30°. Since the number of sides in C is equal to the number of exterior angles, the number of sides $= \frac{360°}{30°} = 12$.

G2. Sketch It if You Can

If possible, sketch each of the following. If it is not possible, then give a reason why it cannot be sketched:

a) A quadrilateral that has exactly one right angle and no parallel sides.

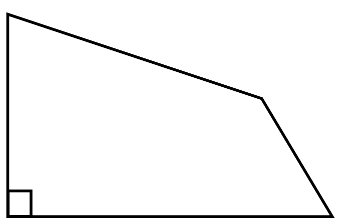
b) A quadrilateral that has exactly two right angles and no parallel sides.

c) A quadrilateral that has exactly three right angles.

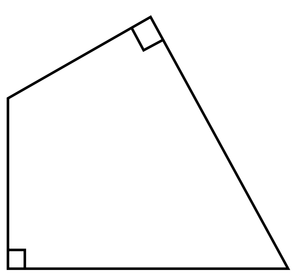
d) A quadrilateral that has exactly one right angle and exactly one pair of parallel sides.

A SOLUTION

a)



b)



c) This figure is impossible.

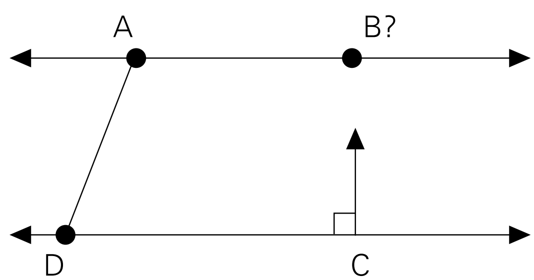
Suppose that a quadrilateral had exactly three right angles. Since the sum of the interior angles of a quadrilateral is 360°, and the sum of the three right angles is $3 \times 90° = 270°$, the remaining angle would be $360° - 270° = 90°$. Thus, if three angles of a quadrilateral are right angles, all four angles must be right angles.

d) This figure is impossible.

Let's try to construct the figure. We'll start with two parallel lines, $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$. The right angle has to be somewhere; we'll put it at C. We extend a ray perpendicular to $\overleftrightarrow{DC}$ to make the right angle. Wherever it hits line $\overleftrightarrow{AB}$, that's where we put point B.

Since ray $\overrightarrow{CB}$ crosses both $\overleftrightarrow{DC}$ and $\overleftrightarrow{AB}$, and $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{DC}$, the angles $\angle$ABC and $\angle$ BCD are supplementary angles. Therefore, $\angle ABC + \angle BCD = 180°$.

But $\angle BCD = 90°$, so $\angle ABC$ must equal 90° also. Thus, any quadrilateral with two parallel sides and one right angle must have at least two right angles.



G3. Name That Common Solid

Here we describe some top, front and side views of familiar solids; that is, cones, cylinders, pyramids and cubes. In each case, identify a solid whose views match the given description, and sketch a view of it that shows its principal features.

a) Side view and front view are triangles. Top view is a circle.

b) Side view and front view are rectangles. Top view is a circle.

c) Side view and front view are triangles. Top view is a square.

d) Side view and front view are triangles. Top view is a rectangle.

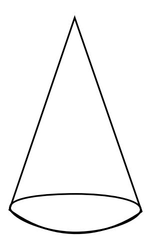
e) Side view and front view are rectangles. Top view is a rectangle.

f) Side view, front view and top view are all congruent squares.

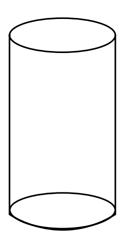
g) Side view, front view, and top view are all congruent, and all triangles.

A SOLUTION

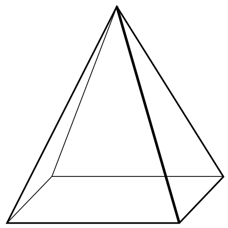
a) Cone



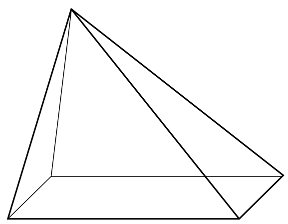
b) Cylinder



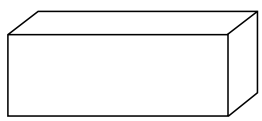
c) Square pyramid



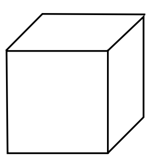
d) Rectangular pyramid



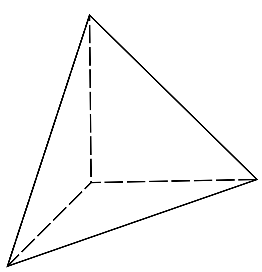
e) Rectangluar prism



f) Cube



g) Triangular pyramid



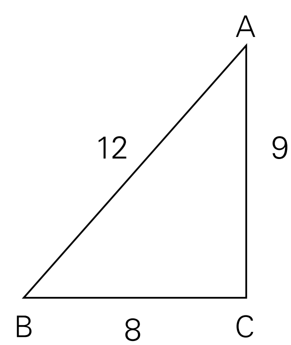
G4. Right or Almost Right?

The sides of a triangle have lengths 8, 9 and 12 centimeters. One of the angles of this triangle is either a right angle or close to it.

Decide whether this angle is exactly a right angle, a little more than a right angle or a little less than a right angle.

A SOLUTION

If ABC were a right triangle, then its longest side must be the hypotenuse. By the Pythagorean theorem, the square of its length, 12^2, or 144, would be the sum of the squares of the lengths of the other two sides, namely $8^2 + 9^2$, or 145. But since $144 < 145$, it follows that $12^2 < 8^2 + 9^2$.

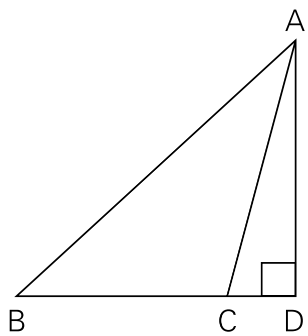


Thus, $\angle ACB$ is not a right angle after all.

Indeed, since $8^2 + 9^2 \approx (12.04)^2$, it appears as if $\angle ACB$ must be a bit smaller than 90°, since the side opposite $\angle ACB$ is a bit shorter than it would be if $\angle ACB$ were a right angle. This is intuitive, but somewhat difficult to prove.

Suppose it were not true. That is, suppose that, $\angle ACB$ is bigger than 90°.

Then if we drop a perpendicular from A, it would meet the extension of BC at D, where $BD = BC + CD$.



By the Pythagorean theorem,

$$AB^2 = BD^2 + AD^2$$

$$= (BC + CD)^2 + AD^2$$

$$= BC^2 + CD^2 + AD^2 + 2(BC)(CD).$$

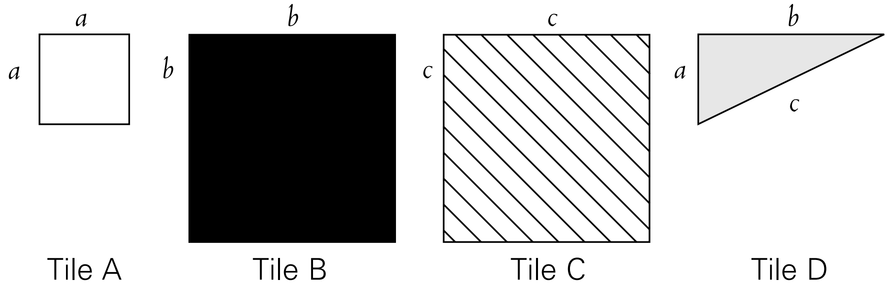
But ADC is also a right triangle, so the Pythagorean theorem also implies that $CD^2 + AD^2 = AC^2$.

By substituting this into the expression for AB^2, we see that: $AB^2 = BC^2 + AC^2 + 2(BC)(CD)$.

This means that $12^2 = 8^2 + 9^2 + 2(BC)(CD)$, (i.e., $144 = 145 + 2(BC)(CD)$), which is impossible since $2(BC)(CD) > 0$, and $144 < 145$.

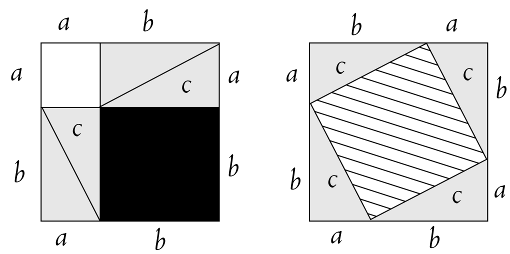
This shows that $\angle ACB$ must be less than 90°.

G5. Tiles



A store sells tiles in four different designs. Three tiles are square and the other is a right-angled triangle.

The tiles will be used to make patterns. Here are two ways they can be fitted together:



a) Explain how you know that the two figures above are squares.

b) Suppose that $a = 10$ cm and $b = 20$ cm. Calculate the area of each tile and the length of side c.

A SOLUTION

a) To conclude that each figure is a square we need to verify that, in each figure, all four sides are straight lines of equal length and all four angles are right angles. By inspection we can see that each corner is a right angle and each side has length $a + b$. So all that we need to prove is that the sides of each figure are straight lines.

Proving that something is straight amounts to showing that the angles along the way are straight angles — measuring 180°.

For the figure on the left, two sides (upper and right) are clearly straight lines, since they are formed by two right angles. The left and lower sides are created from three angles that, differently arranged, are the angles of the triangular tile D. Since the interior angles of a triangle add up to 180°, so do the angles that create those two sides of the figure. Hence both sides are straight lines, so the figure is a square.

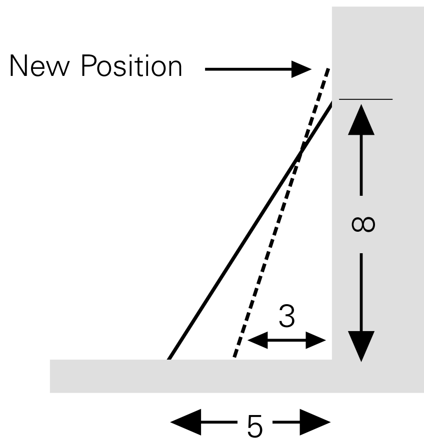
The same argument shows that each side of the other figure is also a straight line. The angles at the point where the tiles come together are the same as the interior angles of triangular tile D. Thus they form a straight line.

b) The area of tile A is 10^2 cm^2 $=$ 100 cm^2, the area of tile B is 20^2 cm^2 $=$ 400 cm^2 and the area of tile D is $(\frac{1}{2})(10)(20)$ cm^2 $=$ 100 cm^2. Thus, the area of tile C can be calculated by subtraction from the composite figure:

$(10+20)^2 - 4(100) = 30^2 - 400 = 900 - 400 = 500$ cm^2.

The area of tile C is 500 cm^2, so the length of a side is $\sqrt{500} = 10\sqrt{5}$.

G6. Leaning Ladders



The diagram shows a ladder leaning against a wall, in such a position that it meets the wall 8 feet from the ground and meets the ground 5 feet from the wall.

If the ladder's bottom is moved closer to the wall, changing from 5 feet to 3 feet away, how far up from its present position will the top of the ladder be?

A SOLUTION

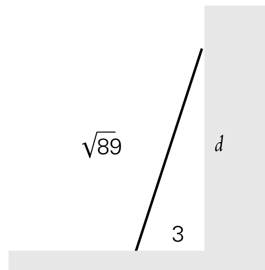
To solve this problem, we need to know the length of the ladder. Since the ladder, wall and ground form a right triangle, we can use the Pythagorean theorem to find the length of the ladder. Let x represent the length of the ladder. Then:

$$5^2 + 8^2 = x^2$$

$$25 + 64 = x^2$$

$$x = \sqrt{89} \text{ feet.}$$

The ladder in its new position can also be modeled by a right triangle, as the diagram below shows.



Let d represent the new height of the top of the ladder above the ground. Then:

$$3^2 + d^2 = 89$$

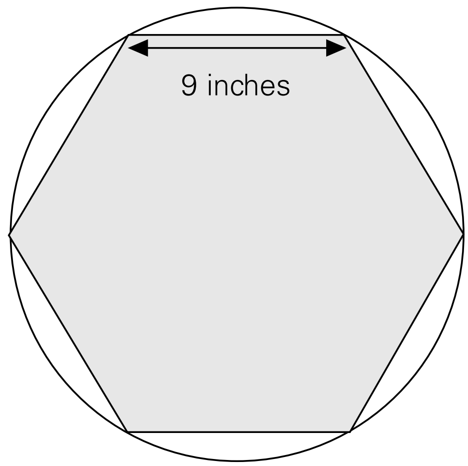
$$9 + d^2 = 89$$

$$d^2 = 80$$

$$d = \sqrt{80}.$$

To find the change in position of the ladder, we subtract the original position of the top of the ladder from the new position. This yields $\sqrt{80} - 8$, which, since $9^2 = 81$, is a little less than 1 foot. $\sqrt{80} - 8 \approx 8.944 - 8 = 0.944$.

G7. Milling Hexagons

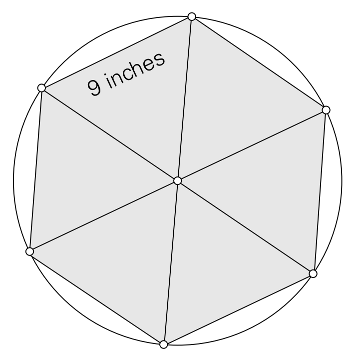


A machinist mills round steel rods so that all cross-sections are regular hexagons, as shown in the diagram above.

One of the hexagonal pieces needs to be 9 inches on a side. What is the diameter of the rod that is needed?

A SOLUTION

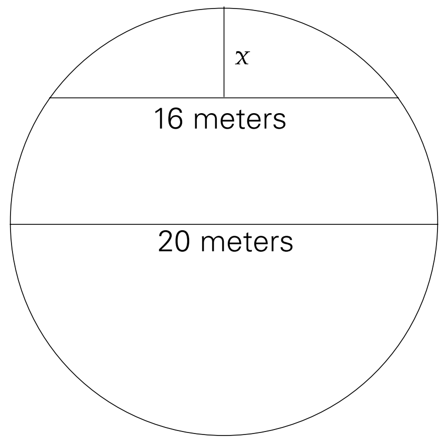
Join the vertices of the hexagon to the center of the circle, as shown.



Because of symmetry, the six central angles are all equal, and each measure $\frac{360°}{6} = 60°$. Since the radii are equal, we know that each of these six triangles is an isosceles triangle.

Therefore, the base angles of each triangle are equal. Since the sum of the angles of a triangle is 180°, the measure of each base angle is $\left(\frac{180° - 60°}{2}\right) = 60°$. This means that the measure of each of the angles of each triangle is 60°, which therefore means that all six triangles are equilateral. This implies that the radius of the rod is 9 inches. Therefore, the diameter is 18 inches.

G8. Chord



The diagram shows a circle with diameter 20 meters that is cut by a chord of length 16 meters.

Find the length x of the segment (shown in the diagram) that is perpendicular to the chord and intersects the chord at its midpoint.

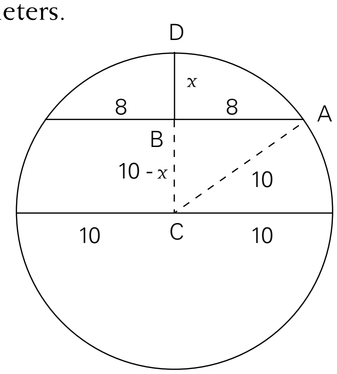
A SOLUTION

Since the diameter of the circle is 20 meters, the radius is 10 meters.

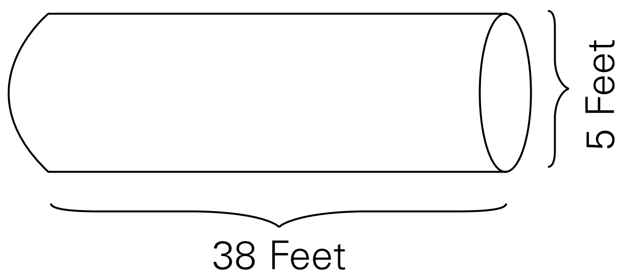
The segment $\overline{BD}$ is the perpendicular bisector of the chord. We can extend it to form a radius of the circle $\overline{CD}$, which has a length of 10 meters. Since B is the midpoint of the chord, $\overline{AB}$ is half the length of the chord: 8 meters.

The radius $\overline{AC}$ is also 10 meters long, and ABC is a right triangle. Therefore by the Pythagorean theorem, $BC^2 + 8^2 = 10^2$, so that $BC = 6$ meters. It follows that $x = DB = CD - BC = 10 - 6 = 4$ meters.

Another way to see this is to observe that ABC is a right triangle with the hypotenuse and a leg equal to 10 and 8. Thus, it is double the size of the common 3-4-5 right triangle, which is a 6-8-10 right triangle. Hence $BC = 6$, so $DB = 4$ meters.



G9. Milk Tanker

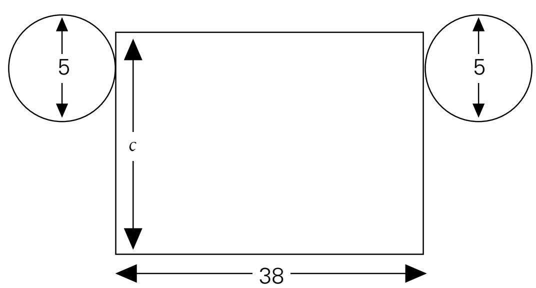


A stainless steel milk tanker in the shape of a right circular cylinder is 38 feet long and 5 feet in diameter.

Determine the amount of stainless steel material needed to construct the tanker, assuming that 12% of the material you start with will be wasted in the construction process.

A SOLUTION

To find the amount of stainless steel material needed to build the tank, we need to first find the surface area of a right circular cylinder that is 38 feet long and 5 feet in diameter (shown in the diagram).



The net for this cylinder would look like the diagram above and have the given dimensions. The height of the rectangle, c, is the same as the circumference of the cylinder base.

Thus:

$c = \quad d = 5 \quad \approx 15.7.$

We can find the surface area of the tanker by finding the total area of the two circles and the rectangle.

$$A_{total} = 2A_{circle} + A_{rectangle}$$

$$= 2 \text{ x } \quad \left(\frac{5}{2}\right)^2 + 38 \text{ x } 5$$

$$= \frac{50}{4} + 190$$

$$= 202.5 \quad \text{square feet.}$$

Since we know that 12% of the materials will be wasted during the construction of the tanker, we can think of the 202.5 ft^2 (approximately 636 ft^2) required at the end of the construction process as 88% of the materials at the start of the construction process.

Let *m* represent the amount of materials needed at the start of construction.

Hence, $(0.88)m = 202.5$

$$m = \frac{202.5}{.88} \quad \approx 723 \text{ square feet.}$$

G10. Two Circles

One circle has a circumference of exactly 22 centimeters. A second circle has a diameter of exactly 7 centimeters.

a) Which circle has a smaller radius?

b) Using the value of on your calculator, find the area of the ring created when the smaller circle is placed inside the larger one. Give the answer to 4 decimal places.

c) Repeat part (b) using the approximation 3.14 for .

d) Repeat part (b) using the approximation 3.1416 for .

e) Repeat part (b) using the approximation $\frac{22}{7}$ for .

A SOLUTION

a) Let's say that circle 1 is the one with the 22-cm circumference, and circle 2 has the diameter of 7 cm.

First, find their radii. For circle 1, $C = 2\ r = 22$, so $r = \frac{11}{\ } \approx 3.5032$

For circle 2, the diameter $= 7$, so $r = \frac{7}{2} = 3.5000$ Thus circle 2 is smaller, but not by much.

b) We could compute the area of circle 1 by using the approximate value r 3.5032, but it is better to use the exact value of $\frac{11}{\ }$. Thus, the area of circle 1 is

$$A_1 = \ \left(\frac{11}{\ }\right)^2 = \frac{121}{\ }.$$

Similarly, the area of circle 2 is $A_2 = \ \left(\frac{7}{2}\right)^2 = \frac{49}{4}$.

The area of the ring between the circles is the difference between these two areas.

$$\begin{aligned} A_{ring} &= A_1 - A_2 \\ &= \frac{121}{\ } - \frac{49}{4}. \end{aligned}$$

Using this formula for the area of the ring and the 8-digit value of   on a calculator, we find that the area of the ring is approximately equal to 0.0310 cm^2.

c) Using 3.14 for , the area of the ring appears to be 0.0700 cm^2.

d) Using 3.1416 for , the area of the ring appears to be 0.0308 cm^2. Note how this is closer to the "calculator" value.

e) Using $\frac{22}{7}$ for , the calculated area of the ring is 0 cm^2:

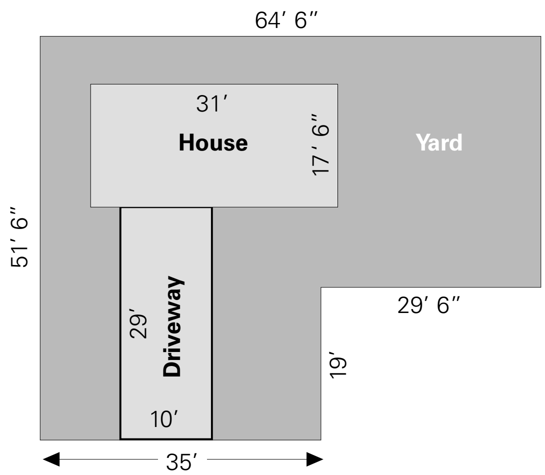
$$\frac{121}{\frac{22}{7}} - \frac{49 \times \left(\frac{22}{7}\right)}{4} = \frac{121 \times 7}{22} - \frac{(7 \times 7) \times \frac{2 \times 11}{7}}{4}$$

$$= \frac{11 \times 11 \times 7}{2 \times 11} - \frac{7 \times 2 \times 11}{4}$$

$$= \frac{11 \times 7}{2} - \frac{11 \times 7}{2}$$

$$= 0.$$

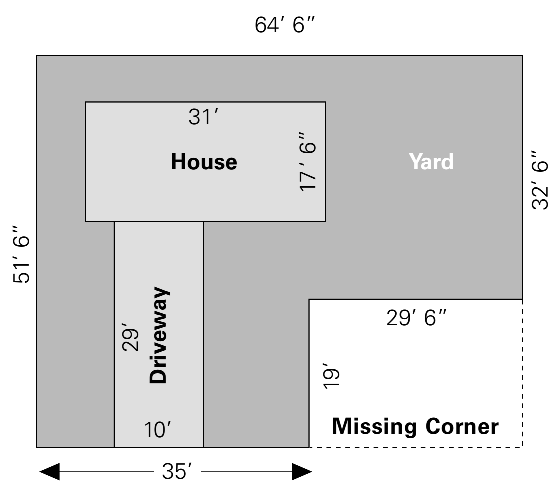
G11. Yard Cover



How many square yards of grass are required to cover the yard above?

A SOLUTION

To find the area of the grass required to cover the yard, we will first find the area of the entire rectangle. Then we'll subtract the portions of the rectangle that are not grass. All areas are measured in square feet.



Area of entire rectangle = 51.5 x 64.5 = 3321.75 ft^2.

Area of driveway = 29 x 10 = 290 ft.

Area of house $= 17.5 \times 31 = 542.5$ ft^2.

Area of missing corner $= 19 \times 29.5 = 560.5$ ft^2.

Area of yard = [area of entire rectangle] – [all the portions that are not grass]

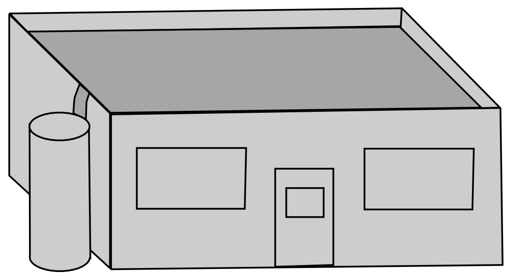
$= 3321.75 - 290 - 542.5 - 560.5$

$= 1928.75$ ft^2.

But we are asked to give the area of the grass in square yards. There are 9 square feet in a square yard. By dividing the area of the grass in square feet by 9, we get the area of the grass in square yards:

$\frac{1928.75}{9} \approx 214.3$ square yards.

G12. Rain



Rain falls on a flat roof that is 22′ by 36′. The rain drains into a cylindrical cistern 12′ high and 5′ in diameter. Assume that all of the rain drains into the cylinder (no evaporation or leaks).

a) How many inches of rainfall will it take to fill the cistern to a depth of 4′ 6″?

b) How many more inches of rain on the roof would cause the cistern to overflow?

A SOLUTION

a) To find the amount of rain needed to fill the cistern to a depth of 4′ 6″, we need to first find the volume of water in the cistern when filled to this depth. If r represents the radius of the cistern and h is the height of the water level in the cistern, the volume is:

$$V = r^2h$$

$$= (2.5)^2(4.5)$$

$$\approx 88.4 \text{ ft}^3.$$

The rain collected from the roof is the same volume as a shallow 22′ x 36′ rectangular basin when filled to a depth of d′. This volume is $(22)(36)d$.

Thus, to find the depth of rain d needed on the roof, we equate these two volumes:

$$(22)(36)d \approx 88.4.$$

Solving for d (which is the depth, in feet, of the rainfall on the roof) we get $d \approx 0.112$ feet ≈ 1.34 inches.

b) To find the volume of additional rain needed to overflow the cistern, we need to find the volume of the remaining space in the cistern, which is $12 - 4.5 = 7.5$ feet tall.

We compute the volume as before.

$$V = \ r^2h$$

$$= \ (2.5)^2(7.5)$$

$$\approx 147.3 \text{ ft}^3.$$

And, as before, we find the depth d of the rain on the roof that would yield this volume:

$$(22)(36)d \approx 147.3.$$

This gives $d \approx 0.186$ ft, or approximately 2.23 inches. More than this amount of rainfall will cause the cistern to overflow.

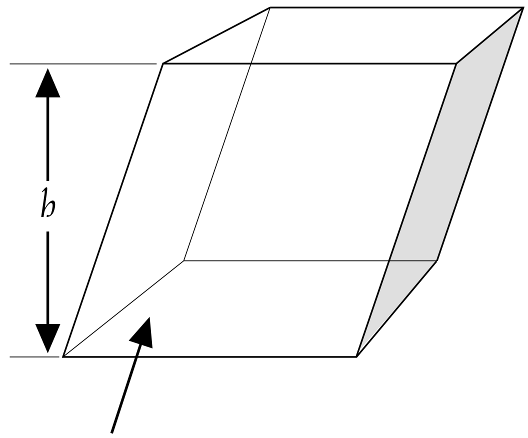
AN ALTERNATE SOLUTION

Since the volume of the cistern is proportional to its height, the extra depth of rain sought is proportional to that calculated in part (a):

$$\frac{d}{1.34} = \frac{7.5}{4.5}$$

Thus, $d = 2.23$ inches.

G13. Prisms



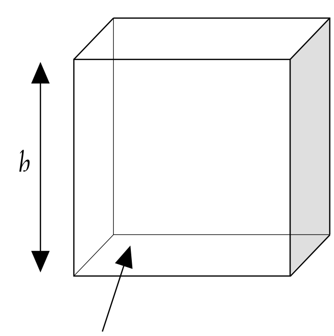
The prism pictured above has height h.

a) Draw a sketch of a different prism with the same base and volume as the prism above.

b) If the area of the base of the prism is 10 square feet, what is the volume of this prism?

A SOLUTION

a) One picture would be a right prism with the same base and the same height. Since volume is just base area times height, the volume remains the same.



b) Volume = [base area] x [height]. We are given that the base area is 10 square feet and the height is h, therefore, the volume is $10h$ in square feet.

G14. Increasing Volume

A rectangular box measures 4.3 centimeters by 6.81 centimeters by 14.007 centimeters. Suppose you can increase just one of these dimensions by exactly 1 centimeter. Which one would you increase if you wanted the largest possible increase in volume?

A BRUTE FORCE SOLUTION

We could solve this first question by trial and error, increasing each dimension by 1 centimeter and finding the dimension that increases the volume the most.

Volume with the original dimensions: $4.3 \times 6.81 \times 14.007 = 410.2 \text{ cm}^3$.

Volume after increasing the first dimension by 1 centimeter: $5.3 \times 6.81 \times 14.007 = 505.6 \text{ cm}^3$.

Volume after increasing the second dimension by 1 centimeter: $4.3 \times 7.81 \times 14.007 = 470.4 \text{ cm}^3$.

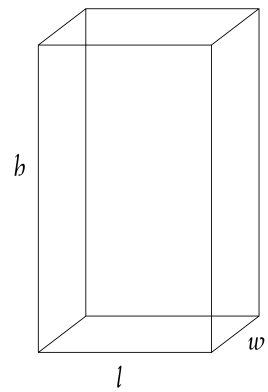
Volume after increasing the third dimension by 1 centimeter: $4.3 \times 6.81 \times 15.007 = 439.4 \text{ cm}^3$.

So increasing the smallest edge length by 1 centimeter results in the largest increase in volume.

A THOUGHTFUL SOLUTION

Although "brute force" calculations such as those above produce an answer, they don't really reveal what's going on. Thinking about it, we see that the additional volume is a (thin) rectangular box. One dimension of this box will be 1 cm, and the other two dimensions will be those of the sides not increased. Since we want the largest possible volume, we want to keep the largest two numbers and increase the smallest one. In other words, add 1 centimeter to the side that is just 4.3 centimeter.

A SYMBOLIC SOLUTION



To gain more "algebraic" insight, we can use the variables l, w and h to represent the length, width and height of the box. The original volume of the box is lwh. If we were to increase one of the dimensions, say h, by 1 centimeter, the volume of the new box would be:

$$V_{new} = lw(h + 1)$$

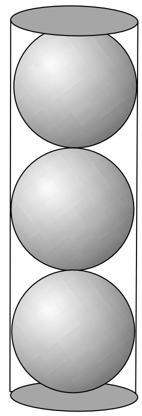
$$= lwh + lw$$

$$= V_{old} + lw.$$

We can see that the new volume is made up of the original volume plus the product of two of the three dimensions. Therefore, to get the largest possible increase in volume, we need to use the largest two dimensions to make the largest product and use the remaining dimension (the smallest edge length) as the dimension we would increase by 1 centimeter.

Therefore, to get the largest increase in volume you should increase the 4.3-centimeter edge length.

G15. Tennis Balls



Tennis balls are often packed snugly three to a can.

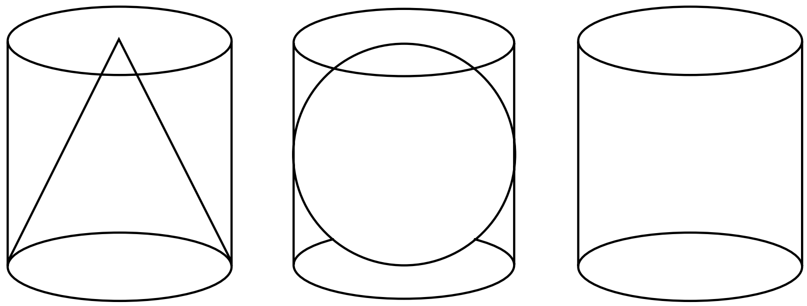
What percent of the volume of the can do the tennis balls occupy?

A SOLUTION

If we let r represent the radius of each tennis ball, then the radius of the can is r, the height of the can is $6r$ and the area of the base is $\;r^2$. Thus the volume of the can is $(6r)(\;r^2) = 6\;r^3$, while the volume of each tennis ball is $\frac{4}{3}\;r^3$. So the percent of the volume occupied by the tennis balls is

$$\frac{3V_{ball}}{V_{can}} = \frac{3 \times \frac{4}{3}\;r^3}{6\;r^3} = \frac{4\;r^3}{6\;r^3} = \frac{2}{3} \approx 67\%.$$

G16. Cone, Sphere, Cylinder



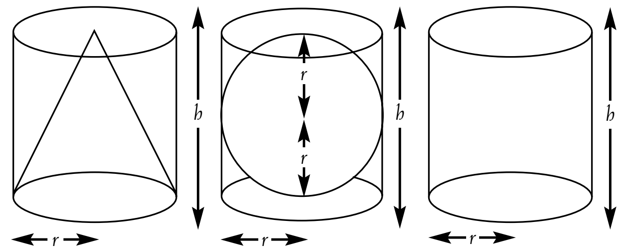
Imagine a cone inscribed in a cylinder of the same size, so that the base of the cone is the same as the base of the cylinder and the top of the cone touches the top of the cylinder. Imagine also a sphere inscribed in a cylinder so that the sphere touches the cylinder at the north and south poles and all the way around the equator.

Show that the ratio Volume of cone : Volume of sphere : Volume of cylinder is $1 : 2 : 3$.

A SOLUTION

Let's use r to denote the radius of the cylinder. This is then the same as the radius of the sphere and the radius of the base of the cone. Therefore, the height of the cylinder is the same as the sphere's diameter, that is, $h = 2r$. This is also the height of the cone. So we'll apply the various formulas for volume and substitute $2r$ for h where appropriate:

Sketching and labeling the cross section makes clear the dimensions of each volume.



The volume of each figure is calculated from the area of the base A and the height h.

$\frac{1}{3}Ah$ $\qquad$ $\frac{2}{3}Ah$ $\qquad$ Ah

In each case $A = \ r^2$ and $h = 2r$.

$\frac{1}{3}\ r^2 \times 2r$ $\qquad$ $\frac{2}{3}\ r^2 \times 2r$ $\qquad$ $r^2 \times 2r$

Simplification yields:

$\frac{2}{3}\ r^3$ $\qquad$ $\frac{4}{3}\ r^3$ $\qquad$ $\frac{6}{3}\ r^3$

To find the ratios, divide by the common factor of $\frac{1}{3}\ r^3$.

2 $\qquad$ 4 $\qquad$ 6

Thus the ratios of the volumes are 1 : 2 : 3.

G17. From the Cross Section

Suppose you know that a box is a rectangular solid with height 20 cm and volume 240 cm^3.

Imagine a horizontal cross section parallel to the base of the box.

a) Can you find the area of this cross section? If so, what is it?

b) Can you find either of the linear dimensions of this cross section? If so, what are they?

Imagine now a vertical cross section parallel to one of the sides.

c) Can you find the area of this cross section? If so, what is it?

d) Can you find either of the linear dimensions of this cross section? If so, what are they?

A SOLUTION

a) The area of the cross section is the same as the area of the base of the box. Therefore, the volume of the rectangular solid, namely (area of base) x height, is the same as (area of cross section) x height. Since we know that the volume is 240 cm^3, we can find the area of the cross section by dividing the volume by the height.

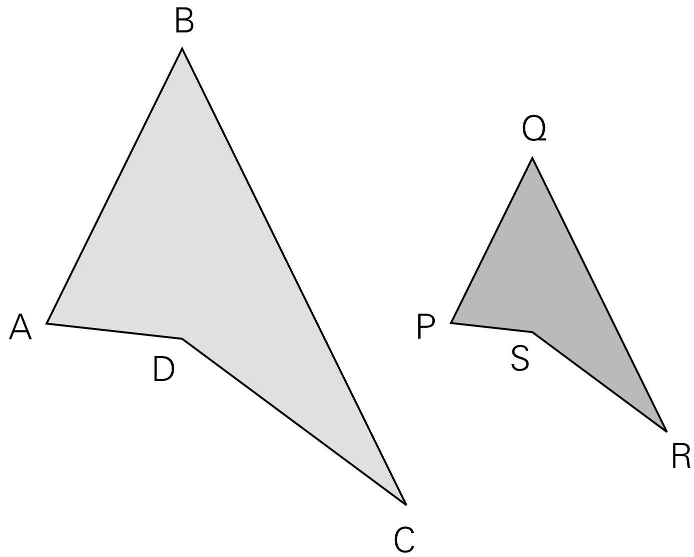
$$\text{Area of the cross section} = \frac{240 \text{ cm}^3}{20 \text{ cm}} = 12\text{cm}^2.$$

b) No. We don't know either linear dimension of the cross section. We only know that their product is 12 cm^2.

c) No. We know that one linear dimension (the height) is 20, but we don't know the other linear dimension we need to find the area.

d) We know one of the linear dimensions (the height) is 20, but we can't determine the other.

G18. Similar Polygons



ABCD and *PQRS* are similar polygons whose perimeters are 40 inches and 30 inches, respectively.

The area enclosed by ABCD is 8 square inches.

a) What area is enclosed by *PQRS*?

b) Is it possible for the straight-line distance from point A to point C to be 20 inches?

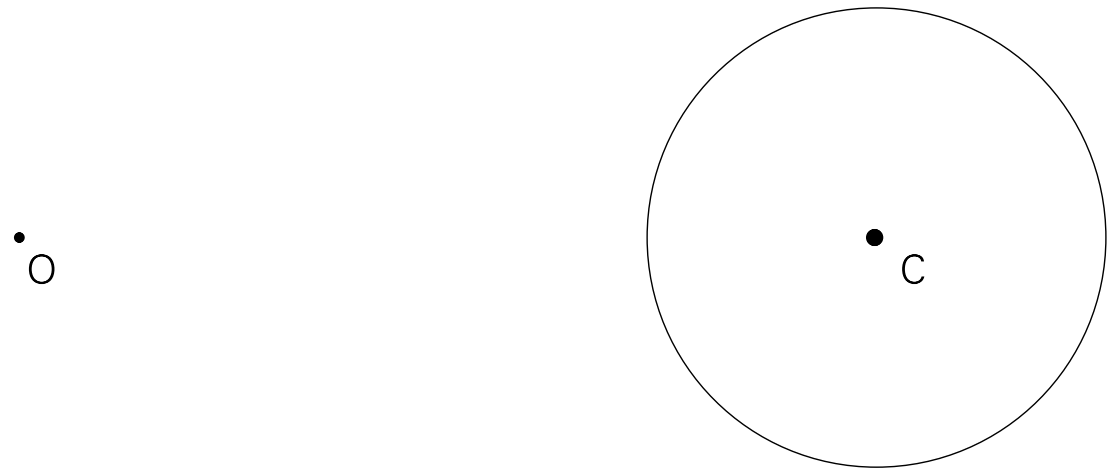
A SOLUTION

a) The perimeter of *PQRS* is 30, and the perimeter of *ABCD* is 40. Therefore the ratio of the linear dimensions of the polygons is $\frac{3}{4}$. So the ratio of the areas of the polygons will be given by the square of the ratio of the linear dimensions, that is, $\left(\frac{3}{4}\right)^2 = \frac{9}{16}$. So:

$$A_{PQRS} = \frac{9}{16} A_{ABCD} = \frac{9}{16} \times 8 = \frac{9}{2} = 4.5 \text{ in}^2.$$

b) No. If the distance from A to C were 20, then by the triangle inequality, $AD + DC > 20$ and $AB + BC > 20$, which would make the perimeter greater than 40.

G19. Circle Transformations



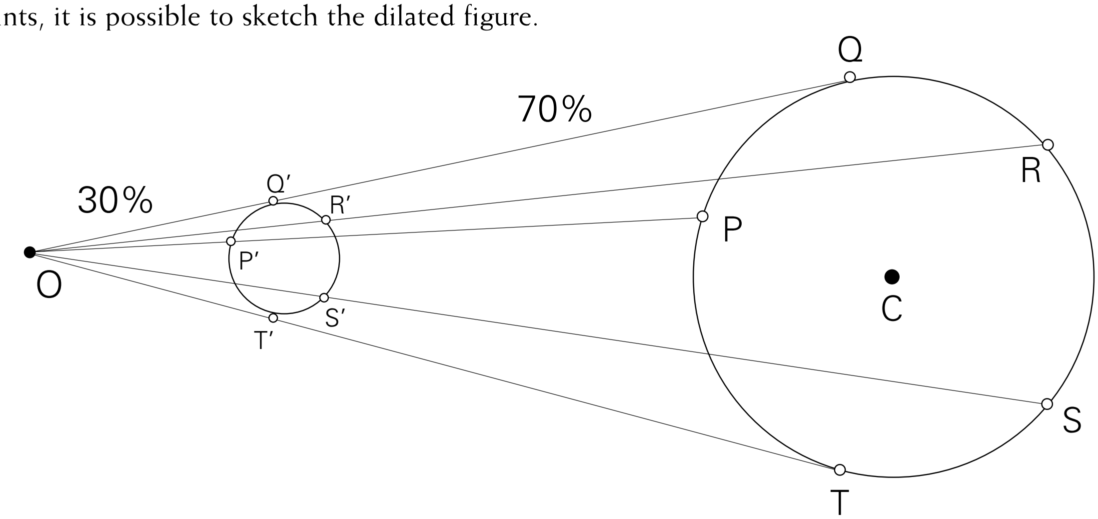
The circle above will be dilated by a factor of 0.3, with the center of dilation being the point O shown.

a) Use a ruler to plot several points on the new image, and then sketch the new image.

b) Will the new image be a circle?

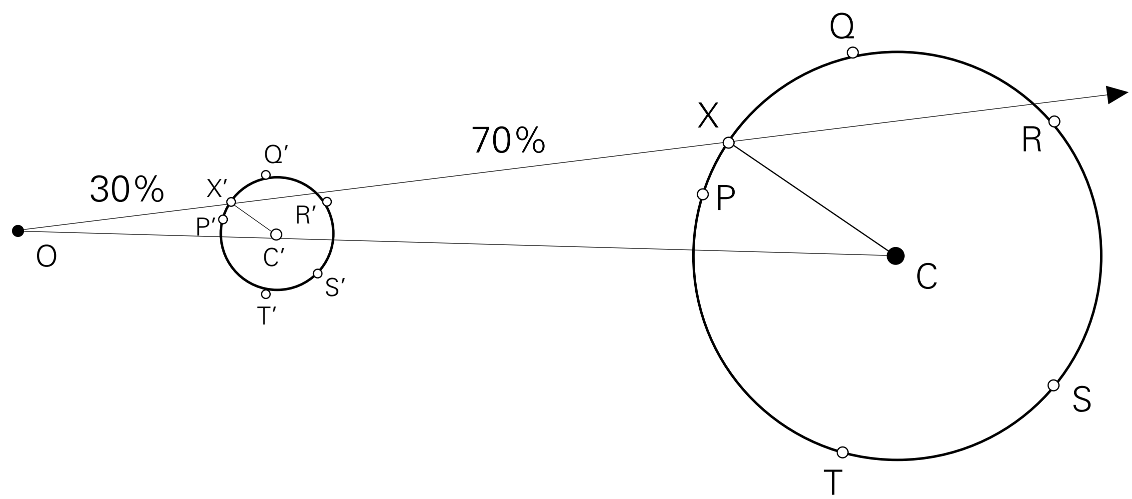
A SOLUTION

a) To form a 0.3 dilation of any figure, use a ruler to connect the center of dilation O with different points on the figure, say P′, Q′, R′, S′, T′, etc. The point P′ that is 30% of the way from O to P along the straight line that connects O with P is a point on the dilated figure. By finding several such points, it is possible to sketch the dilated figure.



b) As we can see, assuming the drawing and measurements are done carefully, the new image appears to be a circle. However the fact that this image is really a circle is not at all obvious. To prove that it is, we need to show that all its points lie equidistant from some one point, which is at its center. (That is, after all, the definition of a circle.)

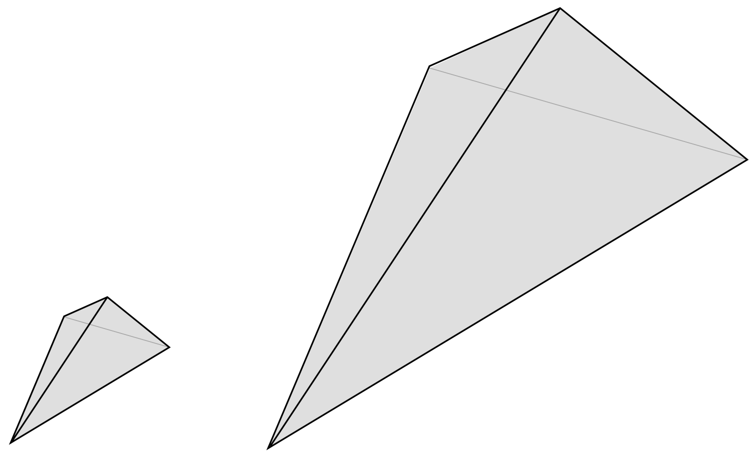
To locate the potential center point of the new image, we join the center C of the original circle to the dilation point O and mark a point C′ that is 30% along the way. Now pick any point X′ on the circumference of the small (dilated) image and draw the line through O that will project this point to a point X on the circumference of the original circle. Look now at the oblique triangles OCX and OC′X′.



Both of these triangles share a common angle at O, and the sides that form this angle are proportional by construction: The sides OC′ and OX′ are exactly 30% of the sides OC and OX, respectively. Thus triangle OCX and OC′X′ are similar, so C′X′ is also 30% of CX.

To summarize: We started with an arbitrary point X′ on the image circle and showed that its distance to C′ is 30% of the distance of the image point X to the center C of the large circle. But since all such distances are equal in the large circle, they must also be equal (but 30% as long) for the image. This shows that the image must be a circle, since all of its points are equidistant from C′.

G20. Two Tetrahedra



The two tetrahedra shown above are similar. The edges of the larger tetrahedron are each three times as long as the corresponding edges of the smaller tetrahedron.

The surface area of the smaller tetrahedron is 48 in^2.

The volume of the smaller tetrahedron is 16 in^3.

a) What is the surface area of the larger tetrahedron?

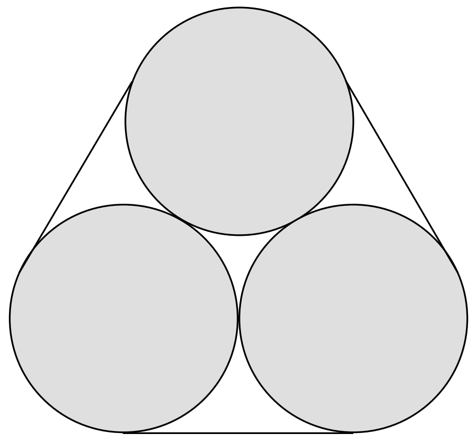
b) What is the volume of the larger tetrahedron?

A SOLUTION

a) The edges of the larger tetrahedron are each three times as long as the edges of the smaller tetrahedron. The ratio of the linear dimensions is 1 : 3. So the ratio of the area is 1 : 9. Hence the surface of the larger tetrahedron is $48 \times 9 = 432$ in^2.

b) The ratio of the volumes of these two similar figures will be the cube of the ratio of their linear dimensions. That is, the ratio of the volumes is 1 : 27. So the volume of the larger tetrahedron is $16 \times 27 = 432$ in^3. (That the numbers are the same is a coincidence that depends on the units.)

G21. Length of Rope

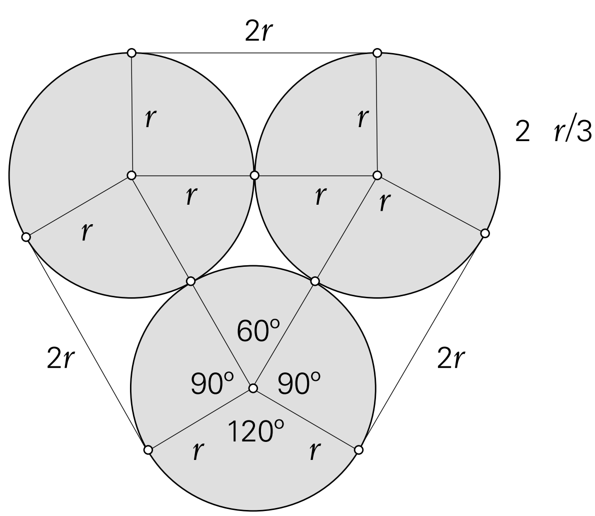


Three cylinders of the same radius *r* are tied together snugly by a rope. The diagram above shows a cross-sectional view.

What is the length of the rope around the cylinders?

A SOLUTION

The details of this solution are shown in the diagram below. The diagram shows that the rope is made up of a section that fits snugly around some portion of each circle and three line segments that do not actually touch the circles (except at points of tangency). The problem here is to find the length of each line segment and the portion in each circle that is snugly surrounded by the rope.

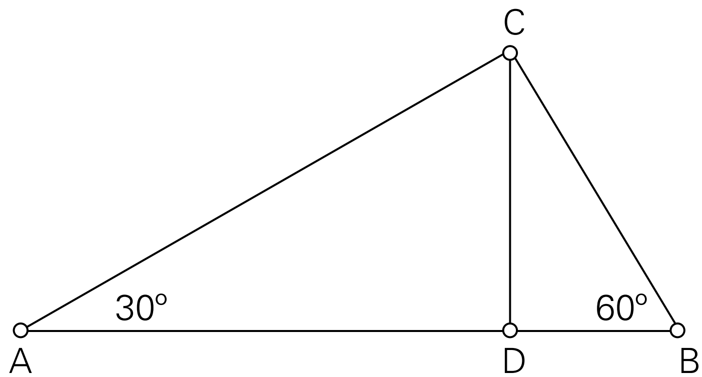


From the diagram, we can see that each line segment begins and ends at a point of tangency. At each of these points of tangency, the line segment and the radius of the circle are perpendicular.

If we join the centers of each pair of circles with a straight line, we can see from the diagram that we have created three rectangles of length $2r$ and width r. The length of each rectangle gives the length of each line segment of rope as $2r$.

Joining the centers of the circles creates an equilateral triangle with each side equal to $2r$. The part of each circle snugly surrounded by rope can be measured by computing the central angle subtended by this part of the circle. Looking at the diagram, we see that this central angle is 120° because it is 360° minus two right angles (coming from two rectangles) and an angle of the equilateral triangle. In other words, $360° - (2 \times 90°) - 60° = 120°$. Thus the part of each circle snugly surrounded by rope is $\frac{120°}{360°} = \frac{1}{3}$ of the circumference. Since there are three such pieces, the total length of these pieces is the same as the circumference of the circle. So the length of the rope consists of the circumference of one circle of radius r plus three lengths each $2r$ units long, that is, $L = 2\ \ r + 6r$.

G22. Right Triangle



Triangle ABC is a right triangle with $\angle$ ACB the right angle and A and B measuring 30° and 60°, respectively. $\overline{CD}$ is perpendicular to $\overline{AB}$.

a) If $\overline{AB}$ has length 4 inches, find the lengths of $\overline{AC}$ and $\overline{BC}$.

b) Find the measurements of $\angle$ACD and $\angle$DCB.

c) List all pairs of similar triangles in the picture.

d) Find the lengths of $\overline{CD}$, $\overline{AD}$ and $\overline{DB}$.

A SOLUTION

a) In a 30-60-90 triangle, the short leg is half the length of the hypotenuse, and the long leg is $\sqrt{3}$ times the short leg. (The sides are in the ratio 1 : 2: $\sqrt{3}$.) So:

$\overline{BC} = \frac{1}{2}\overline{AB} = 2$ inches, and

$\overline{AC} = \sqrt{3}\,\overline{BC} = 2\sqrt{3}$ inches.

b) Let's look at ADC and CBD. We're given one acute angle in each triangle. In addition, we know that the angles at point D, namely $\angle$ADC and $\angle$CDB, are both right angles. So, since the sum of the interior angles in a triangle is 180°,

$$\angle ADC + 90° + 30° = 180°$$

$$\angle ADC = 60°$$

$$\angle BCD + 90° + 60° = 180°$$

$$\angle BCD = 30°.$$

c) Because the corresponding angles are equal, the three triangles in the figure are all similar:
ACD ~ CBD ~ ABC.

d) Since the triangles are similar, the corresponding sides are proportional and in the same proportions as the original large triangle.

So $\overline{DB}$ (the length of $\overline{DB}$, the short leg of the small triangle) is half of $\overline{CB}$ (which was 2). So $\overline{DB} = 1$.

Therefore, $\overline{AD} = 3$ (since $\overline{AB} = 4$).

Finally, $\overline{CD}$ is $\sqrt{3}$ times $\overline{DB}$ (which is 1). So $\overline{CD} = \sqrt{3}$. Or we could observe that $\overline{CD}$ is half of $\overline{AC}$ to obtain the same result.

Sample Problems: IV. Alegbra

Contents

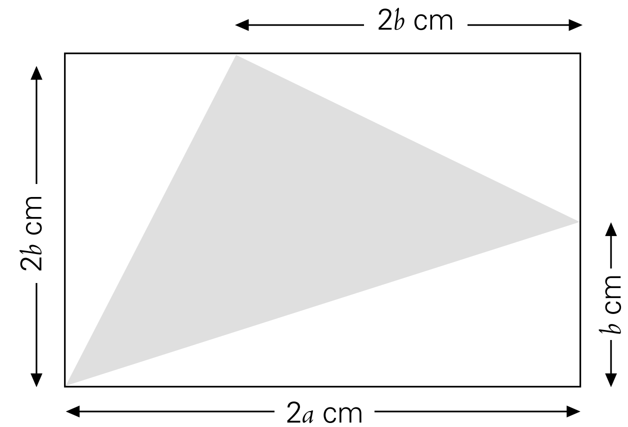
Summary of expectations:

A. Symbols and operations:
- Understand how letters are used to represent numbers
- Know the conventions for writing algebraic expressions
- Generate and use algebraic expressions and equations to solve problems

B. Functions:
- Recognize, represent, interpret and use linear functions in a variety of contexts
- Recognize and represent simple common nonlinear functions
- Translate among different representations of functions, including numeric, verbal, tabular, graphic and algebraic
- Transform and graph quadratic functions by factoring and completing the square
- Understand the relation of quadratic functions to their graphs by expressing the functions in various forms

C. Equations:
- Generate, solve and use single and simultaneous linear equations in one and two variables
- Use graphs to estimate or check solutions to single and simultaneous linear equations in one and two variables
- Recognize that the solutions of a quadratic equation are the zeros, if any, of the corresponding quadratic function

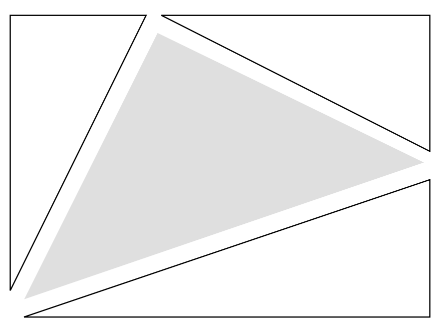
A1. Shaded Triangle



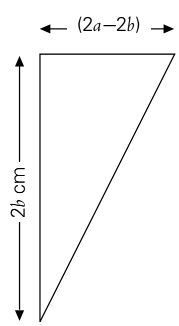
Write an expression for the area of the shaded triangle.

A SOLUTION

Decompose the rectangle into the shaded triangle plus three right triangles:



Top left-hand triangle, area T_1:



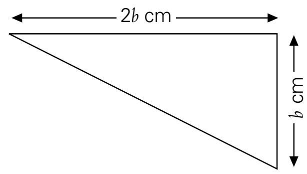
$$\text{Area } T_1 = \frac{1}{2} \times 2b(2a - 2b)$$

$$= 2ab - 2b^2$$

$$= 2b(a - b)\text{cm}^2.$$

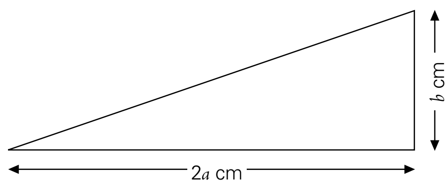
Top right-hand triangle, area T_2:

Area $T_2 = \frac{1}{2} \times b \times 2b = b^2$ cm².



Bottom right-hand triangle, area T_3:

Area $T_3 = \frac{1}{2} \times b \times 2a = ab$ cm².



The area of the rectangle is $A_{rectangle} = 2a \times 2b = 4ab$ cm².

So the shaded triangle's area can be found as follows:

$$T_{shaded} = A_{rectangle} - (T_1 + T_2 + T_3)$$

$$= 4ab - (2b(a - b) + b^2 + ab)$$

$$= 4ab - (2ab - 2b^2 + b^2 + ab)$$

$$= 4ab - (3ab - b^2)$$

$$= ab + b^2.$$

Thus, the area of the shaded triangle is $ab + b^2$ cm².

A2. Adding Odds

Show that the sum of two odd numbers is always an even number.

A SOLUTION

Let the two given odd numbers be represented by $2k + 1$ and $2m + 1$, where k and m are integers.

Then their sum is $(2k + 1) + (2m + 1) = 2(k + m + 1)$, which is even because it is twice an integer.

ANOTHER SOLUTION

Every odd number is one more than an even number (e.g., $9 = 2(4) + 1$), that is, one more than a bunch of pairs. So if two odd numbers are added, the two extra 1s make up an extra pair. Thus, the sum of the two odd numbers is just a larger bunch of pairs; thus, it is even.

A3. Negative or Positive?

a) If a number A satisfies $-5A < 0$, is A positive, negative or 0?

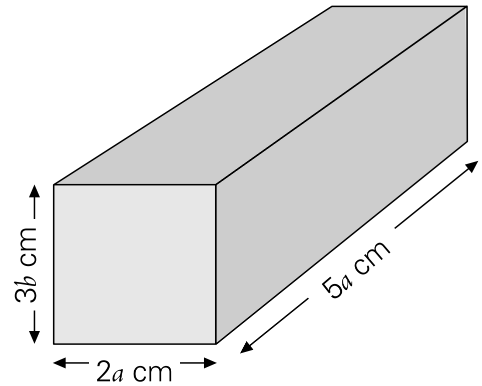
b) If a number B satisfies $2B < 0$, is B positive, negative or 0?

A SOLUTION

a) Suppose that A were negative. Then $(-5) \times (A)$ will be positive. But since $-5A < 0$, A cannot be negative. Suppose now that A is equal to zero. Then $-5 \times A = 0$. Since $-5A < 0$, A is not zero. So A is neither negative nor 0, and therefore A must be positive.

b) Suppose that B were positive. Then $2B > 0$. But $2B < 0$, so B is not positive. Suppose that B is equal to zero. Then $2B = 0$. But $2B < 0$, so B is not equal to zero. Therefore B is negative.

A4. Rectangular Prism



Write an expression for the surface area of a right rectangular prism that has dimensions $5a$, $2a$ and $3b$ centimeters.

A SOLUTION

One way to think of surface area of this rectangular prism is to sketch a net to represent it in two dimensions.

The net shows that the surface area can be decomposed into the area of three pairs of congruent rectangles labeled as R_1, R_2, R_3.

The area of rectangle R_1: $2a \times 3b = 6ab$ cm^2.

The area of rectangle R_2: $2a \times 5a = 10a^2$ cm^2.

The area of rectangle R_3: $3b \times 5a = 15ab$ cm^2.

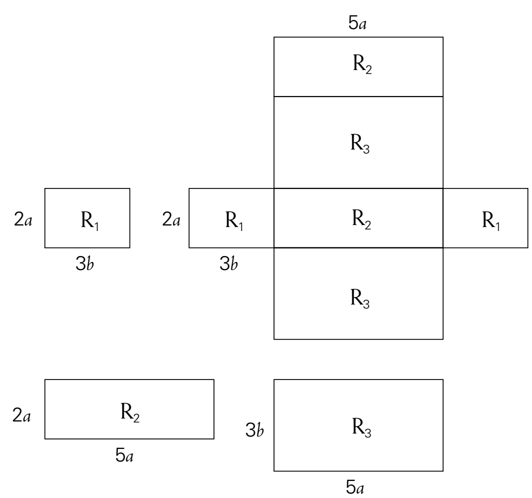
The total surface area is, therefore:

$$A = 2R_1 + 2R_2 + 2R_3$$

$$= 2 \times 6ab + 2 \times 10a^2 + 2 \times 15ab$$

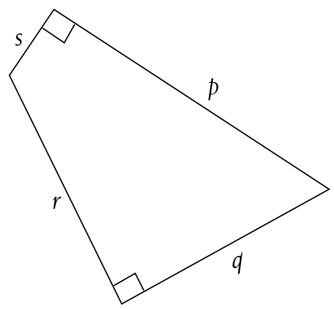
$$= 12ab + 20a^2 + 30ab$$

$$= 20a^2 + 42ab.$$



The total surface area of the rectangular prism is $20a^2 + 42ab$ cm^2.

A5. Perimeter and Area



Write expressions for the area and perimeter of the figure above in terms of the given lengths p, q, r and s as indicated.

A SOLUTION

The perimeter of the quadrilateral is the sum of the lengths of the sides:

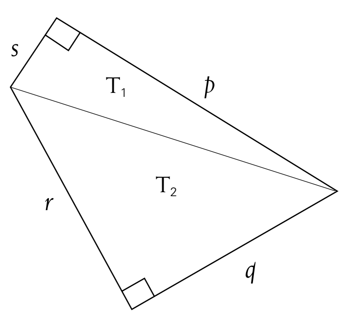
$P_{quadrilateral} = p + q + r + s$ units.

To find the area of the quadrilateral, decompose it into two right triangles T1 and T2.

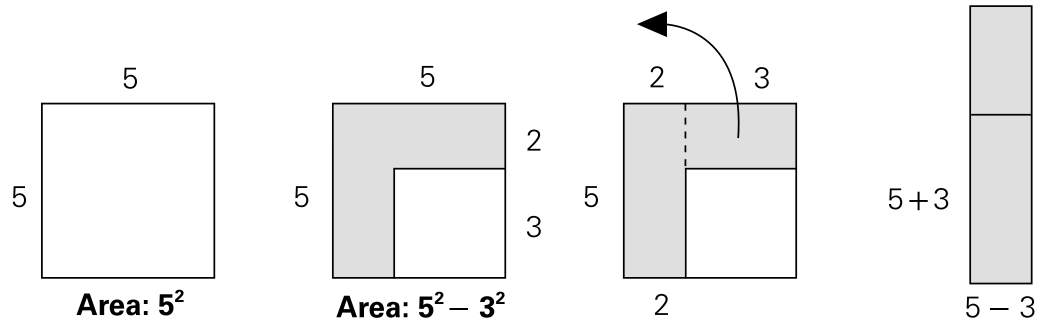
Area $T_1 = \frac{1}{2} \times p \times s = \frac{1}{2} ps$

Area $T_2 = \frac{1}{2} \times r \times q = \frac{1}{2} qr$

Thus, the area of the quadrilateral is given by $\frac{1}{2}(qr + ps)$ square units.



A6. The Difference of Two Squares



The drawings above show how $5^2 - 3^2$ can be rewritten as $(5 - 3)(5 + 3)$.

a) Create a drawing to show that $A^2 - B^2 = (A + B)(A - B)$ for any positive numbers A and B, where $A > B$.

b) Use the result of part (a) to rewrite and evaluate $\left(\frac{7}{2}\right)^2 - \left(\frac{1}{2}\right)^2$.

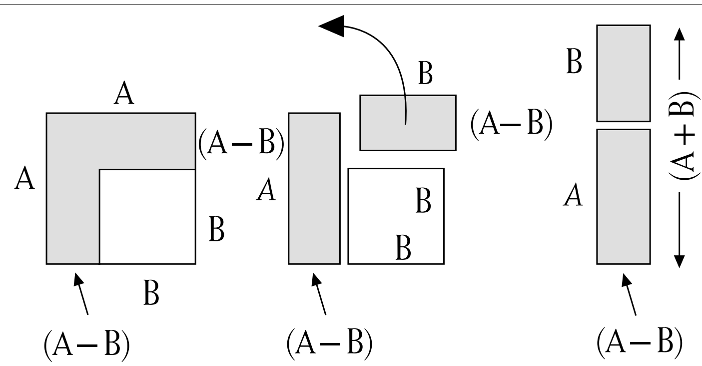
c) Is the formula $A^2 - B^2 = (A + B)(A - B)$ true when $A < B$?

d) Use the result of part (a) to determine which is bigger, 299 x 301 or 300 x 300.

e) Use the result of part (a) to factor 396 into two whole numbers that are close to each other.

A SOLUTION

a)



b) We apply the formula we derived in part (a).

$$\left(\frac{7}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \left(\frac{7}{2} + \frac{1}{2}\right)\left(\frac{7}{2} - \frac{1}{2}\right)$$

$$= 4 \times 3$$

$$= 12.$$

c) The formula is true for any A and B, because $(A + B)(A - B) = A^2 - AB + AB - B^2 = A^2 - AB.$

In particular, it is true for $A < B$, since nothing in this calculation depends on the sign of A or B, or on any relationship between A and B. As an example, let $A = 3$ and $B = 5$. Then $3^2 - 5^2 = (3 + 5)(3 - 5)$ because both sides equal -16.

d) Seeing that we can write the two numbers 299 and 301 as $(300 - 1)$ and $(300 + 1)$ respectively, we can use the strategies of parts (a), (b) and (c) to write:

$$299 \times 301 = (300 - 1)(300 + 1)$$

$$= 300^2 - 1^2$$

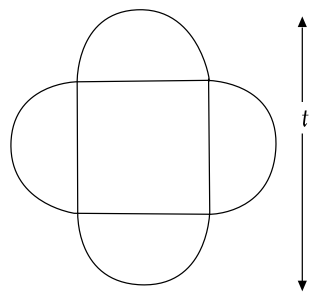
$$= 89{,}999.$$

Hence, 299 x 301 is one less than 300 x 300.

e) Here we first notice that we can write the number 396 as the difference of two squares, namely, $400 - 4$. Thus,

$$396 = 400 - 4 = 20^2 - 2^2 = (20 + 2)(20 - 2) = 22 \times 18.$$

A7. Perimeter and Area Redux

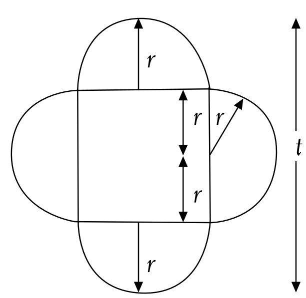


Write expressions for the area and perimeter of the figure above in terms of the given length t as indicated. It consists of four semicircles of equal radius surrounding a square.

A SOLUTION

The perimeter of this shape is made up of four congruent semicircles. To write an expression for the perimeter of this shape we need to express the radius of each semicircle in terms of t.

From this diagram we can see that the height of the figure t is equal to 4 times the radius of one of the semicircles. So $t = 4r$, or $r = \frac{t}{4}$.



The circumference of any circle is $C = 2\ \ r$. This perimeter consists of four semicircles, equivalent to two circles. Therefore, the distance around this figure is equal to 2 x 2 r or 4 r.

Since $r = \frac{t}{4}$, the perimeter is $4\ \ \frac{t}{4} = \ \ t$ units.

The area of this shape is made up of the area enclosed by four semicircles each with radius $\frac{t}{4}$ and the area of the square that they surround. From the diagram, we can see that the side of the square is $\frac{t}{2}$. Therefore, the area of this square is $\frac{t^2}{4}$.

The area enclosed by the four semicircles is equal to the area of one circle enclosed by two circles. The area of one circle is $\ r^2$, so the area of the two circles is $2\ r^2$ where $r = \frac{t}{4}$, giving $2\ (\frac{t}{4})^2 = \frac{2t^2}{16} = \frac{\ t^2}{8}$.

Thus, the total area of this shape is $A = \frac{t^2}{4} + \frac{\ t^2}{8} = \frac{2t^2 + \ t}{8} = \frac{\ +2}{8}\, t^2$ square units.

A8. Fred and Jane

Fred is 3 years older than Jane is.

a) Sketch a graph showing Fred's age as a function of Jane's age for the first 40 years of Jane's life. What is the slope of this graph?

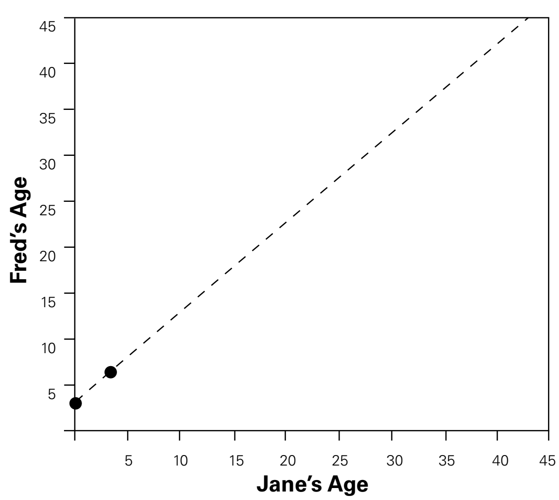
b) Mark the point on the graph that corresponds to when Jane was born.

c) Mark the point on the graph when Fred is twice as old as Jane is.

d) If Fred was twice as old as Jane 10 years ago, how old is each of them now?

A SOLUTION

a) Fred is 3 years older than Jane, so if Jane's age is x, then Fred's age is $x + 3$. The slope of the graph is 1.



b) The vertical intercept (0, 3) represents the point on the graph when Jane was born.

c) The point on the graph when Fred is twice as old as Jane is (3, 6).

d) Ten years ago their ages were (3, 6) so now they are $(3 + 10, 6 + 10)$. Thus, Jane is 13 and Fred is 16.

A9. Tree Circumference

For a certain type of tree, the circumference of the trunk (in centimeters) can be estimated from the age y of the tree (in years) by the formula $C = 2.5y$. Assume that the cross section of the tree trunk is a circle.

a) Estimate the age of a tree that has a circumference of 100 centimeters.

b) Find the radius of the trunk of a 20-year-old tree.

A SOLUTION

a) To estimate the age of a tree that has a circumference of 100 centimeters, we need to solve $C = 2.5y$ where $C = 100$.

$$100 = 2.5y$$

$$y = \frac{100}{2.5}$$

$$y = 40 \text{ years.}$$

Therefore, a tree with a circumference of 100 centimeters is about 40 years old.

b) To find the radius, we first need to find the circumference. To do so we evaluate $C = 2.5y$ where $y = 20$:

$$C = (2.5)(20)$$

$$C = 50 \text{ cm.}$$

The circumference of a tree is given by $2\ \ r$, where r is the radius. To find the radius when we know the circumference, we solve the equation for r:

$$50 = 2\ \ r$$

$$25 = \ \ r.$$

Therefore, the radius is $\frac{25}{\ }$, which is about 8 centimeters.

A10. Pamphlets in the Mail

Different numbers of pamphlets are to be mailed using large boxes that, when empty, weigh 40 ounces each. Each pamphlet weighs 4 ounces. The empty space will be filled with packing pellets. (Since the pellets are very light, you can ignore the weight of the pellets.)

Create a function that represents the relationship between the number of pamphlets in the box and the total mailing weight.

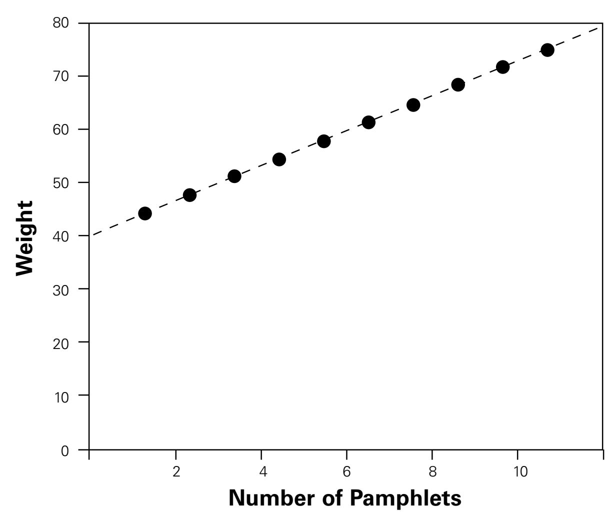
Graph the function, and explain what role the weight of the box and the weight of each pamphlet play in the graph.

A SOLUTION

The weight of an empty box (40 ounces) is the starting point or initial value. For every pamphlet that is added to the box a weight of 4 ounces is added to the mailing weight.

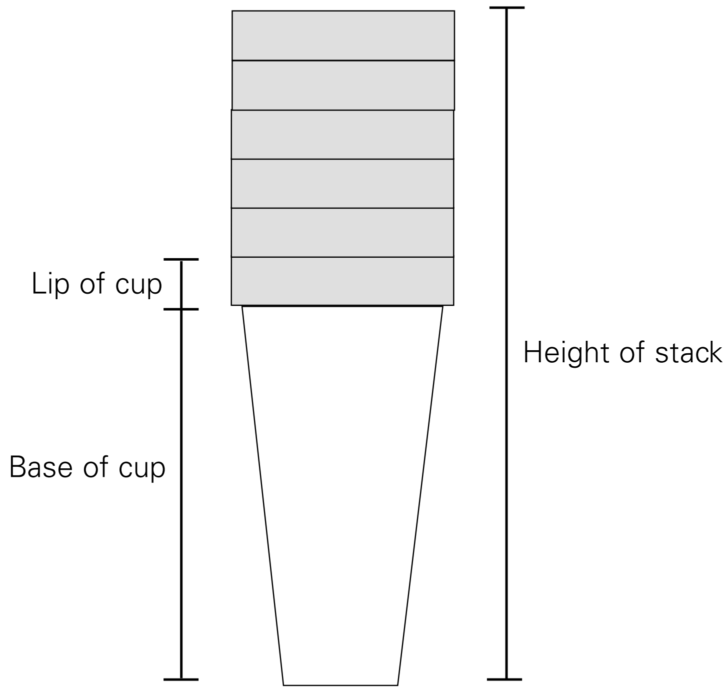
If p is the number of pamphlets to be mailed, then the weight of the pamphlets is $4p$ ounces. Hence, the total weight w of the box and pamphlets together is $w = 4p + 40$.

The graph of this function for $p = 1, 2, 3, \ldots 10$ consists of the points in this graph.



Since $w = 4p + 40$ is a linear function, if we join all the points, we get a straight line as shown. The slope of this straight line is 4, the coefficient of p, and represents the weight in ounces of each pamphlet. The weight of the box, 40 ounces, is the y-intercept of the graph: the weight when $p = 0$.

A11. Paper Cups



The paper cups shown here are identical.

a) By making appropriate measurements, represent the relationship between the number of cups and the height of the stack using a formula and a graph.

b) The graph, which could be a bar graph, can also be drawn as a set of discrete points on a coordinate system. These points can be connected by a straight line. Why is the line straight?

c) What are the slope and intercept of this line? Interpret the meaning of the slope and intercept with regard to the number and size of the cups, or parts of the cups.

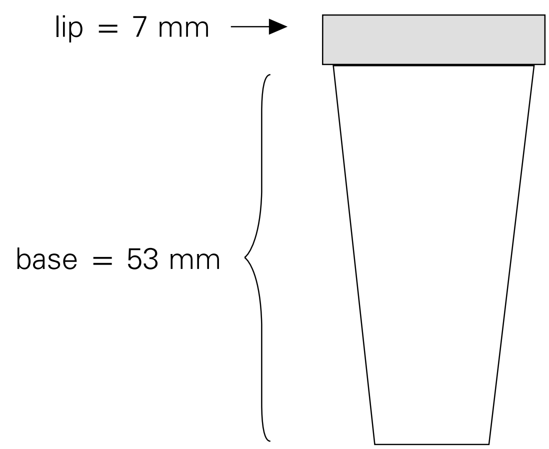
A SOLUTION

a) The measurements in this problem will depend on the size of the paper cup.

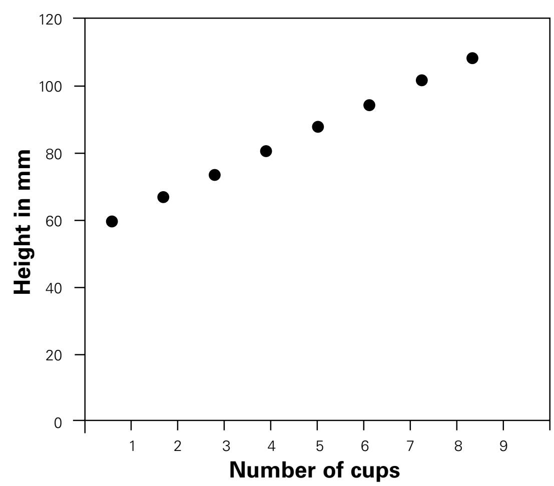
Suppose that the height of the lip of each cup, found by direct measurement, is 7 millimeters, and the base of each cup is 53 millimeters. Therefore, we can think of the height of a stack as consisting of one base (53 millimeters) plus 7 millimeters for each cup lip in the stack.

Let n represent the number of cups in a stack and let h represent the height of a stack. Then $h = 53 + 7n$.

We can represent this function for $n = 1, 2, 3, \ldots$ in the graph shown (h is height in millimeters).



b) All these points lie on a straight line because they satisfy the linear equation $h = 53 + 7n$, and the graph of a linear equation is a straight line.



c) The slope of this line is 7, and the y-intercept is 53. The slope 7 means that each added cup increases the height of the stack by 7 millimeters, which is the thickness of the lip of each cup. On the other hand, the intercept of this line, 53, is not the height of a stack of zero cups, which is why there is no point of the original graph at the y-intercept. Instead, it is the height of a base of a cup without its lip — rather like the grin of the Cheshire Cat.

A12. More Paper Cups

Suppose there is another set of paper cups that stack like those in the preceding problem. The only difference is that these cups are bigger. You are told that the height of one cup (including the lip) is 15 millimeters, and the height of five stacked cups is 28 millimeters.

Find a formula that represents the functional relationship between the height of the stack and the number of cups.

A SOLUTION

Let's retrace the reasoning of the preceding problem, using variables. Let's say the height of the lip is k millimeters and the height of the base is b millimeters. Then, for a total number of n cups, the height of the stack is $h = kn + b$.

If the height of an entire cup is 15 millimeters, then $k + b = 15$, so $b = 15 - k$. Thus,

$$h = kn + b$$

$$= kn + (15 - k)$$

$$= k(n - 1) + 15.$$

We are given that with five cups the total height is 28 millimeters. That is, when $n = 5$, $h = 28$. So,

$$28 = k(5 - 1) + 15$$

$$28 = 4k + 15$$

$$13 = 4k$$

$$k = \frac{13}{4} = 3.25 \text{ mm.}$$

So if $k = 3.25$ and $b = 15 - k$, $b = 11.75$. Therefore, $h = 3.25n + 11.75$. This yields what we were asked to find: a functional relationship between the height of the stack and the number of cups.

Alternate strategy: We know the values of h for $n = 1$ and $n = 5$. Substitute these two sets of values into the formula $h = kn + b$ to get two equations, and then subtract.

$$\begin{array}{r} 28 = 5k + b \\ - \underline{15 = k + b} \\ 13 = 4k \end{array}$$

Now proceed as before, solving for k and using that to find b.

A13. Fruit Punch

John was making fruit punch for a party using crystals that you mix with water. He mixed four scoops of crystals with nine cups of water and it tasted just right. His sister, Sarah, who likes sweet drinks, walked by and dumped another scoop of crystals into the pitcher.

How much water does John need to add so that the fruit punch will taste exactly the same as it did before?

A SOLUTION

Let's find out how much water John's recipe needs per scoop. Let w be the number of cups of water and s be the number of scoops of crystal. For the drink to taste right, we know from John's original mixture that $\frac{w}{s} = \frac{9}{4}$.

But after Sarah adds the additional scoop of powder, $s = 4 + 1 = 5$. In this situation, $\frac{w}{5} = \frac{9}{4}$.

So we solve for w:

$$w = \frac{5 \text{ x } 9}{4} = \frac{45}{4} = 11.25.$$

This means he needs 11.25 cups of water total to make the punch taste just right. Since John originally used nine cups of water, he will need to add $11.25 - 9 = 2.25$ additional cups of water to restore the original flavor.

A14. The Hot Tub

Amy wants to fill up her hot tub for a party. She has one hose that will fill the hot tub in three hours, and she has another hose that will fill it in five hours.

If she turns on both hoses, and if the rate of water flow in each hose remains constant, how long will it take her to fill the hot tub?

A SOLUTION

Let x represent the volume of water coming out of the first hose in one hour and let y represent the volume of water coming out of the second hose in one hour. Then the total volume of water coming out of the first hose in t hours will be tx, and the total volume coming out of the second hose in t hours will be ty.

If v represents the volume of the hot tub, we know from the given information that $3x = v$, and $5y = v$. We need to find the number of hours t it takes to fill the tub with both hoses running at once. In this situation, $tx + ty = v$.

Since $x = \frac{v}{3}$ and $y = \frac{v}{5}$, this means that $\frac{tv}{3} + \frac{tv}{5} = v$ or $t\left(\frac{1}{3} + \frac{1}{5}\right) = 1$.

This yields $t = \frac{15}{8} = 1.875$ hours $= 112.5$ minutes.

ANOTHER SOLUTION

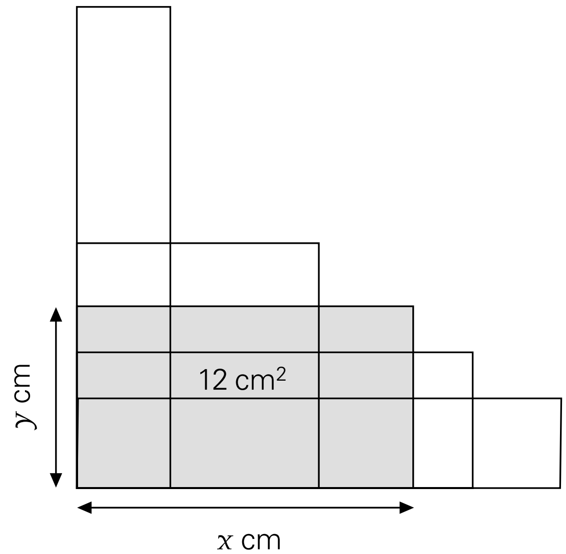
We know that after one hour the first hose will have provided $\frac{1}{3}$ of the water that is needed. Also after one hour, the second hose will have provided $\frac{1}{5}$ of the water that is needed. With both hoses going together, after one hour the tub will hold $\frac{1}{3} + \frac{1}{5} = \frac{8}{15}$ of the required water. So with both hoses going together:

$\frac{8}{15}$ is filled in 60 minutes.

$\frac{1}{15}$ is filled in $\frac{60}{8}$ minutes.

$\frac{15}{15}$ will be filled in $15 \times \frac{60}{8}$ minutes $= 112.5$ minutes, as before.

A15. Length and Width



Let x cm be the width and y cm be the possible lengths of a rectangle with area 12 cm^2.

a) Express y as a function of x.

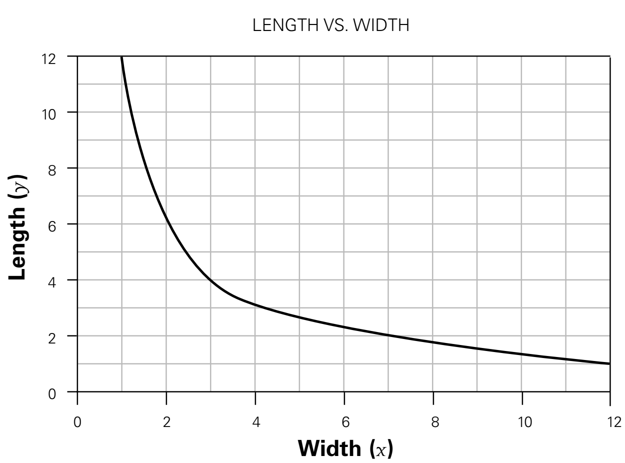
b) Use a graph to represent the relationship between x and y.

A SOLUTION

a) For any rectangle with width x and length y, the area A will be given by: $A = xy$ cm^2.

Here $A = 12$, so $12 = xy$ cm^2. With y as a function of x, we have $\frac{12}{x} = y$.

b)



A16. The Cylinder Company

The Cylinder Company makes right circular cylinders in various shapes and sizes.

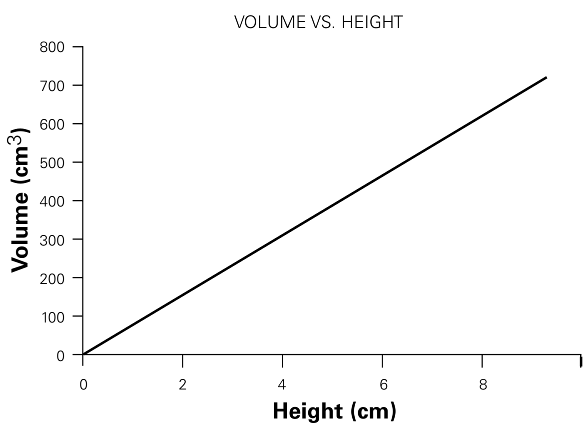
a) The “Model Five” cylinders all have a radius of 5 centimeters. Express the volume of these cylinders as a function of their height. Graph this function.

b) The “Duplo” cylinders all have a height that is exactly twice their radius. Express the surface area of these cylinders as a function of their radius. Graph this function.

A SOLUTION

a) The volume of a cylinder is given by the formula: $V = Ah$ where A is the area of the base and h is the height. Since the area of the circular base is $A = \ r^2$ where $r = 5$, $V = \ (5)^2 h = 25\ h$.

The graph and the formula both tell us that V is a linear function of h.



b) Let S be the surface area of the cylinder. This area consists of three pieces, two circular disks each of area $\ r^2$ that form the top and bottom, and the side whose area is height h times the circumference, $2\ r$, of the base.

Thus,

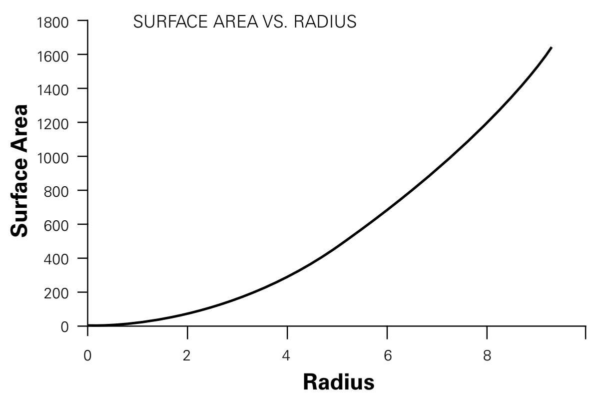
$S = 2\ \ r^2 + 2\ \ rh$

$= 2\ \ r^2 + 2\ \ r(2r)$ (since $h = 2r$)

$= 2\ \ r^2 + 4\ \ r^2$

$= 6\ \ r^2.$

As the formula and the graph show, S is a quadratic function of r.



A17. From the Graph of a Function to the Solution of an Equation

Sketch the graph of the function $(2x + 3)(x - 1)$. That is, sketch the graph of the equation $y = (2x + 3)(x - 1)$.

A SOLUTION

The function $(2x + 3)(x - 1)$, or $2x^2 + x - 3$, is a quadratic, so its graph will be a parabola. Because the coefficient of x^2 is a positive, the graph will be concave up. To sketch a parabola we need to know:

- The intercepts on the x-axis (if any), or, equivalently, the zeros of the function.
- The coordinates of the vertex of the parabola.

Finding the x-intercepts: The graph of $y = (2x + 3)(x - 1)$ will intercept the x-axis at the values of x such that $(2x + 3)(x - 1) = 0$. If the product of two numbers is zero, then at least one of them must be zero. So $2x + 3 = 0$ or $x - 1 = 0$, so $x = -1.5$ or 1. Therefore, the graph passes through the points $(-1.5, 0)$ and $(1, 0)$, which are the x-intercepts.

Finding the coordinates of the vertex of the parabola: Since a parabola has a vertical axis of symmetry, the x-coordinate of the vertex will be halfway between the intercepts, $x = -1.5$ and $x = 1$. The distance between -1.5 and 1 is 2.5. Half of this distance is 1.25, so the point midway between the two x-intercepts is $1 - 1.25 = -0.25$.

Therefore the line $x = -\frac{1}{4}$ is the axis of symmetry.

To find the y-coordinate of the vertex, we evaluate $y = (2x + 3)(x - 1)$ at $x = -\frac{1}{4}$.

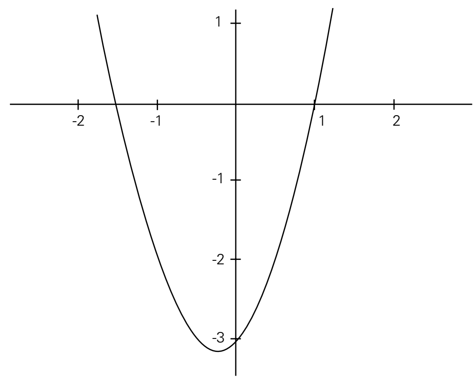
$$y = \left(2\left(-\frac{1}{4}\right) + 3\right)\left(\left(-\frac{1}{4}\right) - 1\right)$$

$$= \frac{10}{4}\left(-\frac{5}{4}\right)$$

$$= -\frac{50}{16} = -\frac{25}{8} = -3\frac{1}{8}.$$

Thus, the coordinates of the vertex are $\left(-\frac{1}{4}, -3\frac{1}{8}\right)$.

The sketch that we are looking for is shown below.



$y = (2x + 3)(x - 1)$

A18. Understanding Quadratics by Completing the Square

a) Show that $f(x) = x^2 - 2x + \frac{3}{4}$ can be rewritten as $f(x) = (x-1)^2 - (\frac{1}{2})^2$.

b) Sketch the graph of $g(x) = (x-1)^2$.

c) On the same diagram, sketch the graph of $f(x) = (x-1)^2 - (\frac{1}{2})^2$.

d) What are the solutions of $x^2 - 2x + \frac{3}{4} = 0$?

A SOLUTION

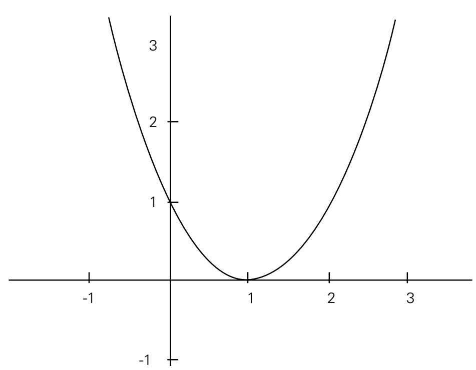
a) $f(x) = x^2 - 2x + \frac{3}{4}$ can be rewritten as follows:

$$f(x) = (x^2 - 2x + 1) - 1 + \frac{3}{4}$$

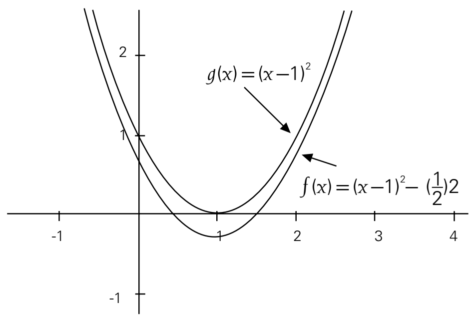
$$= (x-1)^2 - \frac{1}{4}$$

$$= (x-1)^2 - (\tfrac{1}{2})^2.$$

b) The graph of $g(x) = (x-1)^2$ is a parabola that intercepts the x-axis at (and only at) $x = 1$. Hence, the vertex of the graph of $(x-1)^2$ is at $(1, 0)$. To place the parabola on the graph, note that since $g(0) = 1$ and $g(2) = 1$, the points $(0, 1)$ and $(2, 1)$ lie on the graph:



c) The graph of $f(x) = (x-1)^2 - (\frac{1}{2})^2$ is the same shape as the graph of $g(x) = (x-1)^2$, but lower everywhere by $(\frac{1}{2})^2 = \frac{1}{4}$.



d) According to the result of part (a), the solutions for $x^2 - 2x + \frac{3}{4} = 0$ are the same as those of $f(x) = (x-1)^2 - (\frac{1}{2})^2$. So we must solve $(x-1)^2 - (\frac{1}{2})^2 = 0$.

We know that $a^2 - b^2 = (a+b)(a-b)$, so

$(x-1)^2 - (\frac{1}{2})^2 = [(x-1) + \frac{1}{2}]\ [(x-1) - \frac{1}{2}] = 0.$

This occurs when $x - \frac{1}{2} = 0$ or $x - \frac{3}{2} = 0$.

So the solutions of $x^2 - 2x + \frac{3}{4} = 0$ are $x = \frac{1}{2}$ or $x = \frac{3}{2}$.

A19. Completing the Square to Solve a Quadratic Equation

a) Show how you can rewrite $3x^2 - 6x - 4$ as $3\left[(x-1)^2 - \frac{7}{3}\right]$.

b) Show how to use the solution to part (a) to solve $3x^2 - 6x - 4 = 0$.

A SOLUTION

a) First factor out the 3:

$$3x^2 - 6x - 4 = 3\left(x^2 - 2x - \frac{4}{3}\right).$$

Now, inside the parentheses, the first two terms, $x^2 - 2x$, suggest completing the square to get $x^2 - 2x + 1 = (x-1)^2$.

Do this by adding and subtracting 1:

$$\begin{aligned}3x^2 - 6x - 4 &= 3\left(x^2 - 2x - \frac{4}{3}\right)\\ &= 3\left[(x^2 - 2x + 1) - 1 - \frac{4}{3}\right]\\ &= 3\left[(x-1)^2 - 1 - \frac{4}{3}\right]\\ &= 3\left[(x-1)^2 - \frac{7}{3}\right].\end{aligned}$$

b) The calculation in part (a) shows that x is a solution of $3x^2 - 6x - 4 = 0$ if and only if x is a solution of $3\left[(x-1)^2 - \frac{7}{3}\right] = 0$.

So we'll start with that and proceed.

$$3\left[(x-1)^2 - \frac{7}{3}\right] = 0$$

$$(x-1)^2 - \frac{7}{3} = 0$$

$$(x-1)^2 = \frac{7}{3}$$

$$x - 1 = \pm\sqrt{\frac{7}{3}}$$

$$x = 1 \pm \sqrt{\frac{7}{3}}.$$

A20. Solving Equations

Here are three equations:

i) $3x + 5 = 20 - x$

ii) $3x - 5 = 3\left(x - \frac{5}{3}\right)$

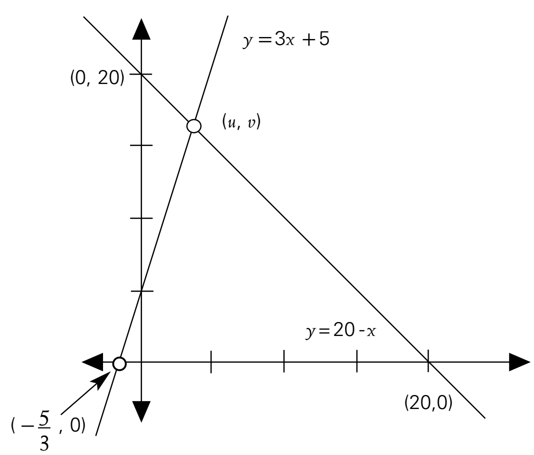
iii) $3x + 5 = 3x + 10$

For each equation, graph the linear functions represented by the left-hand side and the right-hand side. (For example, in (i), graph $y = 3x + 5$ and $y = 20 - x$ on the same coordinate system.) Then, describe how the graphs you made can help you find solutions to the equation. For each equation, list or describe all the solutions, and tell how you know you have them all.

A SOLUTION

Equation (i):

For the equation given in (i), the graphs of $y = 3x + 5$ and $y = 20 - x$ appear below. They intersect at a point that we denote by (u, v).



Since (u, v) lies on both lines, $v = 3u + 5$ and $v = 20 - u$. Setting these equal to each other, $3u + 5 = 20 - u$. If we solve for u, we will find the x-coordinate of the point of intersection — and the solution of the original equation:

$$3u + 5 = 20 - u$$

$$4u = 15$$

$$u = \frac{15}{4}.$$

Substituting $\frac{15}{4}$ for x in the original equation gives $3(\frac{15}{4}) + 5 = 20 - (\frac{15}{4})$, or $\frac{65}{4} = \frac{65}{4}$. So $x = \frac{15}{4}$ is a solution. Since two lines can intersect in at most one point, there can be no more solutions.

The solution of the equation can also be obtained directly as follows (the same algebra, no graphical reasoning):

$$3x + 5 = 20 - x$$

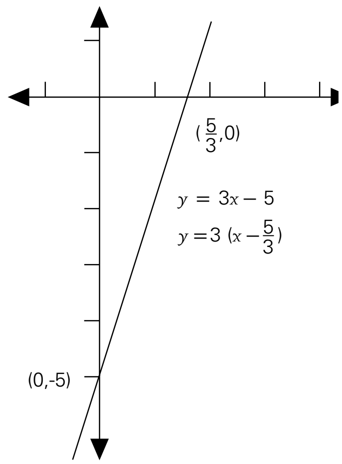
$$4x + 5 = 20$$

$$4x = 15$$

$$x = \frac{15}{4}.$$

Equation (ii):

For the equation given in (ii), the graphs of $y = 3x - 5$ and $y = 3\left(x - \frac{5}{3}\right)$, shown below, coincide:



This shows that it is not possible to find a unique solution of the equation $3x - 5 = 3\left(x - \frac{5}{3}\right)$. Every value of x yields a solution.

If we look at it algebraically, we'd get:

$$3x - 5 = 3\left(x - \frac{5}{3}\right)$$

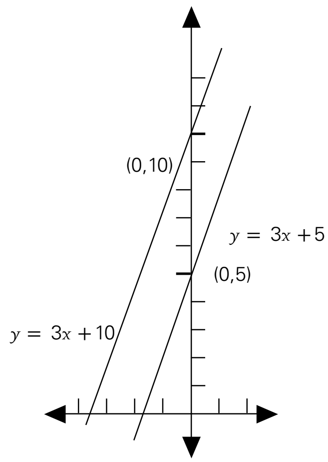
$$3x - 5 = 3x - 5$$

$$3x = 3x$$

$$0 = 0.$$

This result, $0 = 0$, can be hard to interpret. The graph helps us see what it means, namely that the original equation is true for all values of x.

Equation (iii):
For the equation given in (iii), the graphs of $y = 3x + 5$ and $y = 3x + 10$ are shown below.



Since these graphs are parallel, they have no point of intersection. Thus, it is not possible to find a solution of the equation $3x + 5 = 3x + 10$.

Algebraically, if there were a solution, we would get:

$3x + 5 = 3x + 10$

$5 = 10.$

This is impossible, so there must have been no solution.

A21. Calling Plans

Long-distance company A charges a base rate of \$5 per month, plus 4 cents per minute that you are on the phone. Long-distance company B charges a base rate of only \$2 per month, but they charge you 10 cents per minute used.

How much time per month would you have to talk on the phone before it would save you money to subscribe to company A?

A SOLUTION

We can create cost functions to represent each of these calling plans.

Company A has a base rate or initial charge of \$5 plus 4 cents per minute. A cost function in minutes for company A is $C = 500 + 4m$ where m represents the number of minutes used. (Note: We have chosen cents for our units, hence the "500." We could have used $C = 5 + 0.04m$. But an easy mistake is to use $C = 5 + 4m$.)

Company B has a base rate or initial charge of \$2 plus 10 cents per minute. A cost function in minutes for company B is $C = 200 + 10m$ where m represents the number of minutes used.

So if we set these two cost functions equal, we get the following equation:

$$500 + 4m = 200 + 10m.$$

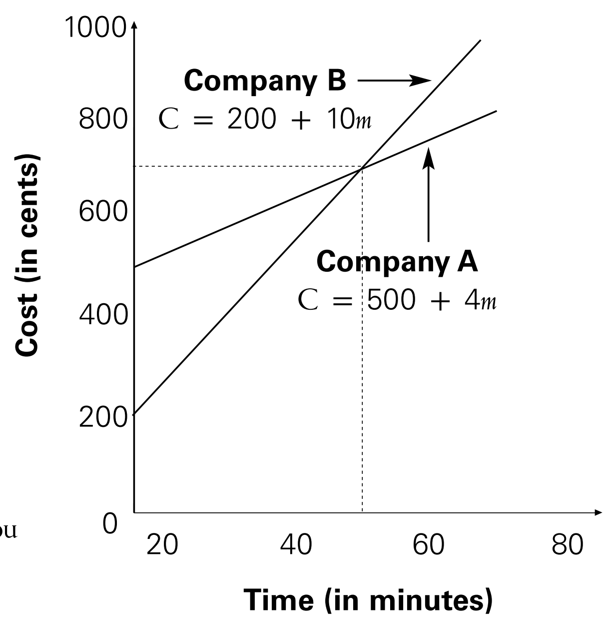
If we solve this equation, we find the number of minutes for which both plans cost the same amount of money.

$$4m + 500 = 10m + 200$$

$$300 = 6m$$

$$m = 50.$$

At 50 minutes the costs are equal, so company A saves you money after 50 minutes. Here is a sketch of the situation:



A22. Consecutive Integers

a) Suppose three consecutive integers have a sum of 195. If x denotes the middle of these numbers, write an equation in x that can be used to find them.

b) Show that the sum of any three consecutive integers is always a multiple of three.

A SOLUTION

a) If the middle number is x, then the first number is $(x - 1)$ and the last number is $(x + 1)$. We can write an equation for the sum and solve it.

$$(x - 1) + x + (x + 1) = 195$$

$$3x = 195$$

$$x = 65.$$

So the three numbers are 64, 65 and 66.

b) As above, if x is the middle of the consecutive numbers, the sum S of all three is $S = (x - 1) + x + (x + 1) = 3x$, which is a multiple of 3.

Suppose that we started by naming the first integer n (rather than the middle one x). Then the algebra will be a bit different, but the result will be the same:

$$S = n + (n + 1) + (n + 2)$$

$$= 3n + 3$$

$$= 3(n + 1).$$

And since n is an integer, so is $(n + 1)$, and the sum, $3(n + 1)$, is a multiple of 3.

A23. Two Cylinders

Two cylinders, A and B, share some dimensions. The height of cylinder A is the same as the radius of cylinder B, and the height of cylinder B is the same as the radius of cylinder A.

a) Cylinder B has exactly twice the volume of cylinder A. What is the relation of the height to the radius of cylinder B? Of cylinder A?

b) Draw these two cylinders approximately to scale.

A SOLUTION

a) The volume of cylinder B is twice that of cylinder A. Let a be the height and b be the radius of cylinder B. Then b is the height and a the radius of cylinder A.

$$V_B = \ \mathrm{r}^2 b = \ b^2 a$$

$$V_A = \ \mathrm{r}^2 b = \ a^2 b$$

$$V_B = 2V_A$$

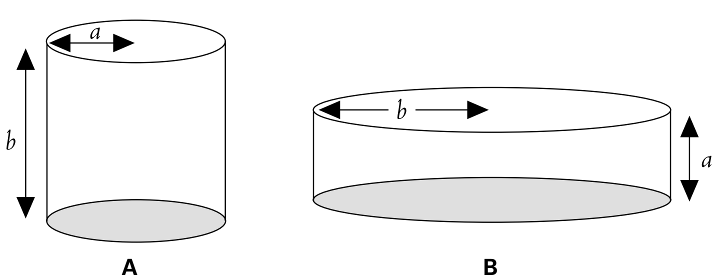
$$b^2 a = 2 \ a^2 b$$

$$(\ ab)b = (\ ab)2a$$

$$b = 2a.$$

For cylinder B, the radius is twice the height. For cylinder A, the height is twice the radius.

b) Perhaps the cylinders would look like this:



A24. Pencils and Erasers

At a certain store, three pencils and two erasers cost \$1.26. At the same store, six pencils and three erasers cost \$2.19.

How much is each pencil, and how much is each eraser?

A SOLUTION

Let p represent the cost of a pencil and e the cost of an eraser.

Then, according to the given information:

(i) $3p + 2e = 126$

(ii) $6p + 3e = 219$.

If we multiply equation (i) by 2, we can then subtract equation (ii) from the resulting equation:

$$\begin{array}{rr} & 6p + 4e = 252 \\ - & \underline{6p + 3e = 219} \\ & e = 33. \end{array}$$

Thus, one eraser costs 33 cents.

Substitute the value of $e = 33$ into equation (i):

$$3p + 2(33) = 126$$

$$3p + 66 = 126$$

$$3p = 60$$

$$p = 20.$$

So one pencil costs 20 cents.

A25. No Solution

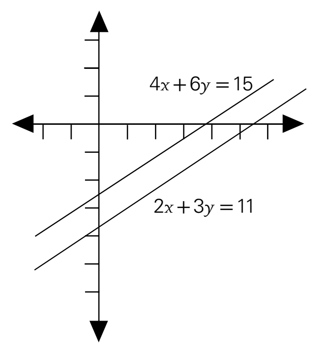
The following pair of linear equations in two variables has no solution. Use graphs and algebra to show why this is the case.

$2x - 3y = 11$

$4x - 6y = 15.$

A SOLUTION

The graph shows that the lines represented by $2x - 3y = 11$ and $4x - 6y = 15$ are parallel. For this pair to have a solution, the lines would need to intersect.



We can use algebra to explore why this pair does not have a solution. Multiply both sides of the top equation, $2x - 3y = 11$, by 2 to get $4x - 6y = 22$.

Now subtract $4x - 6y = 15$ from $4x - 6y = 22$:

$$\begin{array}{rl} 4x - 6y &= 22 \\ -\underline{4x - 6y} &\underline{= 15} \\ 0 &= 7. \end{array}$$

This contradiction shows that there is no solution to the pair of equations, since if there were any numbers x and y that satisfied the equations, we would be forced to conclude that $0 = 7$.

Instead of reasoning toward a contradiction, we can use algebra also to show that the lines are parallel. First, we solve the original equations for y. This gives two equations in slope-intercept form.

$$2x - 3y = 11 \qquad 4x - 6y = 15$$

$$y = \frac{2}{3}x - \frac{11}{3} \qquad y = \frac{4}{6}x - \frac{15}{6}$$

$$y = \frac{2}{3}x - \frac{5}{2}$$

Since the slopes are the same $(\frac{2}{3})$, the lines are either parallel or identical. But since the y-intercepts are different, they must be two different lines that, being parallel, never cross.

A26. Test Scores

Thirty students in a class took a quiz with five questions. On this quiz the class average score was 3.2. The individual scores are as follows:

Scores:	0	1	2	3	4	5
Number of Students:	2	1	?	?	9	4

How many students got 3 points? How many got 2?

A SOLUTION

Let n be the number of students who scored 2, and m be the number who scored 3. Then,

$$n + m = 30 - (2 + 1 + 9 + 4)$$

$$= 30 - 16$$

$$= 14.$$

Thus $m = 30 - n$.

Now the total of all scores will be:

$$T = (2 \times 0) + (1 \times 1) + (2 \times n) + (3 \times m) + (4 \times 9) + (5 \times 4)$$

$$= 2n + 3m + 57.$$

But since the average score is the total score divided by the total number of students, this sum is also:

$$T = 3.2 \times 30 = 96.$$

Combining the last two equations, and substituting $m = 14 - n$ from above, we get:

$$
\begin{aligned}
2n + 3m + 57 &= 96 \\
2n + 3(14 - n) &= 96 - 57 \\
-n + 42 &= 39 \\
n &= 3 \\
m &= 11.
\end{aligned}
$$

So the number of students who scored 2 is 3, and the number who scored 3 is 11.

A27. The Road Trip

Jennifer made a 100-mile trip by car on an interstate highway that was being repaired. She began at noon and arrived at her destination at 3:30 p.m. Because of the construction zone, she was able to average only 20 mph for a long stretch in the middle of the trip. The rest of the time she drove 70 mph.

How many miles long was the construction zone?

A SOLUTION

Let x denote the number of miles of construction (during which Jennifer drove 20 mph); then $100 - x$ will be the number of miles when she drove 70 mph. The total time of the trip, 3.5 hours, consists of the time she was driving 20 mph, namely $\frac{x}{20}$, plus the time she was driving 70 mph, namely $\frac{(100 - x)}{70}$. Thus:

$$\frac{x}{20} + \frac{(100 - x)}{70} = 3.5.$$

To simplify, multiply both sides by 20 X 70 = 140:

$$7x + 200 - 2x = 490$$

$$5x = 290$$

$$x = 58.$$

So, the length of the construction zone was 58 miles.

A28. Number Triangle

Create a number triangle by listing consecutive odd numbers as shown below, with each row having one more number than the preceding one:

Row 1:	1
Row 2:	3 5
Row 3:	7 9 11
Row 4:	13 15 17 19
Row 5:	21 23 25 27 29
Row 6:	
Row 7:	

a) Complete Rows 6 and 7.

b) What is the first number in Row 20? What is the last number in Row 20? [Hint:The n^{th} odd number is $2n - 1$.]

c) Complete the following pattern:

Total of Row 1: 1	$= 1$	$= 1^3$
Total of Row 2: $3 + 5$	$= 8$	$= 2^3$
Total of Row 3: $7 + 9 + 11$	$= 27$	$= 3^3$
Total of Row 4: $13 + 15 + 17 + 19$	$= 64$	$= 4^3$
Total of Row 5:		
Total of Row 6:		

What do you think is the sum of all the numbers in Row 10? In Row 20? In Row n?

d) Using the information from step 3, complete the following table:

Entries	Total	Mean
Row 1: 1	1^3	$\frac{1^3}{1} = 1^2$
Row 2: 3 5	2^3	$\frac{2^3}{2} = 2^2$
Row 3: 7 9 11	3^3	$\frac{3^3}{3} = 3^2$
Row 4: 13 15 17 19	4^3	$\frac{4^3}{4} = 4^2$
Row 5:		
Row 6:		
Row 7:		

Describe in words the pattern that you see. What do you conjecture is the mean of Row 20?

e) Compare the row means with the number of each row and the first number in each row. Describe the pattern that you see. From this pattern, can you guess a formula for the first number in each row?

f) Can you prove that your guess is correct?

A SOLUTION

a) By just following instructions for constructing the number triangle, we get:

Row 5: 21 23 25 27 29

Row 6: 31 33 35 37 39 41

Row 7: 43 45 47 49 51 53 55

b) According to the way the number triangle is defined, if T is the total number of numbers in the first 19 rows, then the first number in Row 20 will be the $(T + 1)^{st}$ odd number. So we first need to find T, and then find the $(T + 1)^{st}$ odd number.

The number of numbers in each row increases by one each time a row is added. So the total number T of numbers in the first 19 rows is:

$T = 1 + 2 + 3 + 4 + \ldots + 17 + 18 + 19.$

If we repeat the same series in reverse order,

$T = 19 + 18 + 17 + \ldots + 4 + 3 + 2 + 1,$

it is easy to see that the total of both series together is

$2T = (1 + 19) + (2 + 18) + (3 + 17) + (18 + 2) + (19 + 1),$

which is just 20 added up 19 times, or 380. Since this is twice T, we find that T, the total number of numbers in the first 19 rows, is $\frac{380}{2} = 190$.

So the first number in Row 20 will be the 191st odd number, which is $2 \times 191 - 1 = 381$.

Since there are 20 numbers in Row 20, the last number will be the 210th odd number, or $2 \times 210 - 1 = 419$.

c) The pattern for sums of the Rows is:

Sum of Row 4: $13 + 15 + 17 + 19 \quad = 64 = 4^3$

Sum of Row 5: $21 + 23 + 25 + 27 + 29 \quad = 125 = 5^3$

Sum of Row 6: $31 + 33 + 35 + 37 + 39 + 41 \quad = 216 = 6^3$

Sum of Row 7: $43 + 45 + 47 + 49 + 51 + 53 + 55 \quad = 343 = 7^3$

So it appears as if the sum of all the numbers in Row 10 is 10^3, the sum of Row 20 is 20^3 and the sum of Row n is n^3. To check this conjecture for Row 20, we can add it all up (using the information above to tell us what the first and last numbers are in this row). Notice that by pairing the numbers as shown, we have 10 pairs, each with a sum of 800.

$381 + 383 + 385 + \ldots + 415 + 417 + 419$

$= (318 + 419) + (383 + 417) + (385 + 415) + \ldots + (399 + 401)$

$= 800 + 800 + 800 + \ldots + 800 = 8000 = 20^3$.

d) Entries	Total	Mean
Row 1: 1	1^3	$\frac{1^3}{1} = 1^2$
Row 2: 3 5	2^3	$\frac{2^3}{2} = 2^2$
Row 3: 7 9 11	3^3	$\frac{3^3}{3} = 3^2$
Row 4: 13 15 17 19	4^3	$\frac{4^3}{4} = 4^2$
Row 5: 21 23 25 27 29	5^3	$\frac{5^3}{5} = 5^2$
Row 6: 31 33 35 37 38 41	6^3	$\frac{6^3}{6} = 6^2$
Row 7: 43 45 47 49 51 53 55	7^3	$\frac{7^3}{7} = 7^2$

Since the row sums are the cube of the row number, and the number of entries in each row equals the row number, the mean of each row will be the square of the row number.

Thus, the mean of Row 20 will be $20^2 = 400$, and the mean of the n^{th} row will be n^2.

e) Using information from above, we can begin the following table:

	Number of Entries	First Term	Row Mean
Row 1:	1	1	1
Row 2:	2	3	4
Row 3:	3	7	9
Row 4:	4	13	16
Row 5:	5	21	25
Row 6:	6	31	36
Row 7:	7	43	49

It appears from this table that the number of entries in each row (n) plus the first term is one more than the mean (average) of the row, which we just discovered is n^2. Said differently, the first term appears to be $n^2 - n + 1$.

f) Our conjecture is that the first term in row n is $n^2 - n + 1$. To show that this conjecture is correct, we imitate the calculations used to create the table above.

We know that the first term in each row is the (T + 1)st odd number, where T is the total number of entries in all the previous rows. As the following simple table shows, the $(T + 1)^{st}$ odd number is $2(T + 1) - 1$:

T:	1	2	3	4	5	6	7	8 …	T	(T+1) …
2T:	2	4	6	8	10	12	14	16 …	2T	2(T+1) …
2T− 1:	1	3	5	7	9	11	13	15 …	2T − 1	2(T+1) − 1 …

Thus the first term in each row is $2(T+1) - 1 = 2T+1$.

Now T, the total number of entries in rows 1 through $n - 1$, is given by:

$1 + 2 + 3 + \ldots + (n - 3) + (n - 2) + (n - 1)$.

Using our earlier trick of repeating the series backwards and adding the two series together, we see that

$$T = \frac{1}{2}\left[(1 + (n-1)) + (2 + (n-2)) + (3 + (n-3)) \ldots ((n-1) + 1)\right]$$

$$= \frac{1}{2}\left[n + n + n + \ldots n\right] \quad (n-1 \text{ times})$$

$$= \frac{1}{2}n(n-1).$$

This the first number in row n is:

$2T + 1 = 2\left[\frac{1}{2}n(n-1)\right] + 1 = n(n-1) + 1 = n^2 - n + 1$, thus proving the conjecture.

Appendix A

Mathematical Topics for Grades 1–5

To be well prepared to meet these expectations, students should enter the sixth grade with a robust command of elementary school mathematics, as outlined below.

Number

- Understand the relationship between numbers and quantities.
- Understand the role of place value in writing whole numbers.
- Understand and fluently perform the basic arithmetic computations with integers, decimals and fractions.
- Understand that fractions and decimals are two different representations of the same concept, and be able to convert among equivalent forms of the same number.
- Understand the concept of the number line and the location on it of integers, fractions, mixed numbers and decimals, both positive and negative.
- Understand the concept of percent, and find the percent equivalents of decimals and fractions.
- Understand factorizations, including prime factorization, of small whole numbers.
- Use arithmetic operations on integers and simple fractions to model and solve problems.
- Understand and compute positive integer powers of non-negative integers.
- Use estimation strategies in computation and problemsolving.
- Understand and estimate with very large and very small numbers.

Algebra and functions

- Develop number sentences for problem situations.
- Recognize and use the commutative, distributive and associative properties.
- Recognize graphs as expressing functional relations.

Measurement and geometry

- Use common and nonstandard units to measure objects.
- Select appropriate units for a given measurement task.
- Identify and classify common geometric figures (e.g., polygons, spheres, pyramids, cones and cylinders).
- Understand from physical models simple volume and area relationships among geometric figures.
- Use two-dimensional coordinate grids to represent points and to graph straight lines.
- Understand the concepts of perimeter, area and volume, and use the appropriate units when performing measurements.

Data analysis

- Measure and count a wide variety of physical objects.
- Organize and interpret numerical and categorical data.
- Construct simple graphs and charts from tables of data.
- Calculate and understand the meaning of mean, median, mode and range of numerical data.

Appendix B

Mathematical Topics for Grades 9–11

Foundations for Success: Mathematics Expectations for the Middle Grades provides a foundation for substantive mathematics in high school that would leave all students well prepared for college and careers. The following topics might be included in grades 9–11 as part of a program for most students regardless of future careers. These topics could be sequenced or integrated according to local priorities; they could be followed in grade 12 with any of a variety of courses, including statistics, calculus and computer science.

Linear problems

Perpendicular lines (negative reciprocal slope, orthogonal vectors); lines and planes in three dimensions; and intersection of two planes as a line and of three planes as a point.

Quadratics and other polynomials

Arithmetic and factoring of polynomials; completing the square in quadratic equations; quadratic formula; graphs of quadratic and cubic polynomials; and maxima, minima and zeroes.

Exploring data

Summarizing and interpreting graphical displays of single-variable data; comparing distributions of single variable data; and exploring two-variable data and two-way tables.

Geometry

Properties of parallel lines and transversals, circles, tangents and inscribed angles; equations of circles, relation to Pythagorean theorem, equations of tangent lines; congruence and similarity; and polygons and their properties, sum of interior and exterior angles.

Algebraic solutions

Solution by various methods of systems of two- and three-linear equations; intersection of two lines in the plane and two planes in space; relation of degenerate systems to parallel lines and planes; systems of linear inequalities; equalities and inequalities involving absolute value; and direct and inverse proportion.

Probability

Relative frequency; independence; conditional probability; standard deviation; normal distribution; central limit theorem; and computer simulations.

Straightedge and compass constructions

Perpendicular bisectors; lines tangent to a circle; parallel lines; angle bisectors; regular polygons; and approximation to using perimeter of regular polygon, partition of a line segment into n equal parts.

Surveys and experiments

Design of surveys; role of randomness in sampling; sources of bias; limitations of observational studies; and statistical significance.

Transformations in two and three dimensions

Translations; simple rotations in two and three dimensions using matrices (noting that rotations in three dimensions fix a line); reflections through a line in two dimensions and through coordinate planes in three dimensions; and scaling factors (dilations).

Circular functions

Trigonometric functions, their graphs and simple identities; relation to circular motion and connection with triangle trigonometry; Argand diagram and DeMoivre's formula; and relation to complex numbers.

Functions, curves and analysis

Absolute value and greatest integer functions; exponential functions and common applications; logarithm as inverse function; curves of general quadratics ($Ax^2 + Bxy + Cy^2 + Dx + Ey = F$); and intersections with lines in the plane.

Appendix C

Explanation of Mathematical Subtleties

As the middle school curriculum moves from arithmetic to more advanced mathematics, teachers regularly will encounter subtle issues that have the potential to cause considerable confusion. Here are some terms and issues that the MAP Mathematics Advisory Panel found sufficiently unclear to warrant special explanation.

Generality

Teachers and students often seem to believe that mathematicians primarily are interested in the most general possible statements. This is a considerable misunderstanding of mathematics. General statements are given in mathematics only when they apply to a large number of important cases. Generalization or abstraction for its own sake should not be encouraged. In particular, it usually is better to offer students examples of how a statement is used rather than to focus instruction on the general form of the statement.

For example, a geometric representation of a large square, which has a small square cut out of one corner, is commonly used to introduce the formula $a^2 - b^2 = (a + b)(a - b)$. Formal verification by algebraic means typically comes next, along with explicit examples showing how the formula can provide efficient strategies (but certainly not the only ones) for certain calculations (e.g., $52 \times 48 = 50^2 - 2^2 = 2500 - 4 = 2496$). Effective instruction will give equal weight to all three aspects — geometric representation, algebraic verification and numerical illustration — and not emphasize any one aspect over another.

Equality

One notable source of difficulty in teaching mathematics is the many different ways in which the equal sign is used. Despite what many believe, mathematical symbols are at times ambiguous. For instance, the same equal sign, "$=$", can be used:

- to state a *definition*. In this instance, "$=$" means "replaces" or "can be replaced by." For example, $A = \quad r^2$, $A = bh$, $y = 2x - 1$, or $\quad r^2 = A$, $bh = A$, $2x - 1 = y$. In each of these expressions, the equal sign says that the symbol A or y can be used in place of the expression on the other side of the sign, or, conversely, one can replace A or y by the expression on the other side.
- to represent *identity*. In this instance, "$=$" means "is the same as," as in "two real numbers a and b satisfy exactly one of the following three relations: $a < b$, $a = b$, $a > b$."
- to represent *equivalence*. In this instance, "$=$" means "represents the same object as." The most common example is the notion of equivalent fractions, e.g., $\frac{3}{4} = \frac{6}{8}$. In literal meaning, $\frac{3}{4}$ represents three copies of one-fourth of the whole, while $\frac{6}{8}$ represents six copies of one-eighth of the whole. These are distinct. When we call them equal, we actually have made a significant logical jump from literal objects to what mathematicians call "equivalence classes."
- to express a *condition*. In this instance, "$=$" establishes an equation that conveys a condition on its variables. For example "$=$" in the equation $x^2 - 3x + 1 = 2x + 1$ establishes a condition on x that is satisfied only when x is 0 or 5. Here the condition on x conveyed by the equal sign can

be thought of just as well as asserting the equality of the two functions $f(x) = x^2 - 3x + 1$ and $g(x) = 2x + 1$.

Interpretation of the equal sign must be derived from context. For students who are just learning, this is often very confusing. To be entirely logical, we should use different symbols for each distinctly different meaning. Indeed, computer science uses the symbol "$:=$" to signify equality with the first meaning, while the third meaning is represented in advanced mathematics by the equivalence symbol "$\equiv$." But tradition in school mathematics is to use a single symbol for these several different meanings. Consequently, the MAP expectations follow common practice and use the same symbol for all four meanings. However, careful attention to distinguishing among these different uses during instruction almost certainly will be worthwhile.

Equations

Often in school mathematics, one hears teachers and students talk about manipulating equations (in the fourth meaning of equality). This is done by adding or subtracting equal quantities from both sides of the equation, or by multiplying or dividing both sides by the same expression. That each of these steps creates a new equation often is overlooked. None of these manipulations replaces or modifies the original equation, which remains unchanged. The hope is that each new equation will have the same solutions as the previous equation. In simple cases, this wish is fulfilled. But some manipulations (e.g., squaring both sides) produce an equation that may have more solutions than the original equation.

Linearity

The study of mathematics begins with linear phenomena both because of their inherent simplicity and because linear functions are the first approximation to common nonlinear phenomena. For this reason, the MAP expectations for middle school mathematics are dominated by the study of linear functions. When a linear function $y = ax + b$ has no constant term, the variables x and y are called "proportional," and the study of this special case ($y = ax$) is often referred to as "proportional reasoning." The equation $y = ax$ arises in many different contexts (for example, in studying similar geometric figures, motion under constant speed or revenues in relation to sales). Through this one equation, therefore, students may begin to perceive the unity of mathematics even on an elementary level.

Proportionality

Proportional relationships are at the heart of quantitative understanding of the world, and their representation in mathematics is of central importance. Yet the treatment of proportionality in middle school textbooks often is vague and confusing. One source of confusion is the tradition of locating "proportionality" as a topic within the number strand as if it were a property (like primeness) of numbers when, in fact, it is a fundamental concept of algebra, geometry and data analysis. A second confusion arises by misappropriation of the term "linear" as a synonym for "proportional." For example, the relation $y = 3x + 2$ is linear but not proportional, whereas $y = 3x$ is both linear and proportional. Finally, in many classrooms and curriculum guides, the clear concept of a proportional relationship between variables often morphs into muddy discussion of "proportional reasoning," as if this were some special kind of inference.

Historically, proportionality has been approached by studying the equality of two ratios. Alternatively, and increasingly in middle school classrooms, it can be approached through the study of linear functions. Both interpretations are important for middle school, but the two must be connected strongly.

Quadratics

In *Foundations for Success*, the quadratic function and its graph are considered to be far more fundamental than the quadratic equation — which all too often is presented in textbooks as a canned technique topped off with the quadratic formula — something to be memorized and generally forgotten after the exams. The strategy in *Foundations for Success* is to suppress the actual formula but include all that leads up to its development. In eighth grade, examples should be specific and numerical. Literal coefficients such as a, b and c should wait until the ninth grade, where their study would be new and challenging. Presentation should begin with factorable cases of increasing complexity, then lay the foundation for completing the square through specific examples. Each step can be reinforced with graphing exercises that build through examples an intuitive understanding of the relation between the coefficients and zeros (roots) of the equation and the position and shape of the associated parabolic graph.

What People Are Saying About the Mathematics Achievement Partnership

"We are in a crisis educationally, and we need to make improvements more rapidly — especially in areas like mathematics. MAP is designed to help states and schools accelerate their improvement efforts. For example, each state now uses its own test to measure student performance. That makes no sense. With the new eighth-grade assessment that MAP is developing, we'll know our efforts are gauged against the best in the world. And states will be saving money by working together and learning from each other — and not trying to reinvent the wheel."

Roy Romer
Superintendent, Los Angeles Unified School District

"Intel is committed to support efforts to strengthen middle school math curricula and instruction to help our students achieve at world-class levels. MAP is designed to help states and school districts reach that goal. Intel is very pleased to be a national sponsor of MAP, and particularly to sponsor California's participation in the program. We have high hopes that this endeavor will significantly benefit both teachers and students in the country's most populous state and the rest of the nation."

Craig R. Barrett
President and CEO, Intel

"It is imperative that states pool their intellectual and political capital to fashion the improvements we all must make to raise achievement in our schools. The [MAP] math partnership is a model for collaboration."

John Engler
Governor, State of Michigan

"We cheat American eighth graders if all we expect of them is what they themselves call 'dummy math,' while students around the world get algebra and geometry. … These are the foundation skills our kids need to succeed in college and at work. These math expectations, agreed upon by the states, and the teaching and testing tools that will be developed from them will, at long last, give our children a chance at a world-class math education."

Louis Gerstner, Jr.
Chairman and CEO, IBM Corporation

"Achieve recognizes that to improve student performance, it is necessary to align curricula, classroom lessons, professional development and assessment with rigorous mathematical goals. MAP is putting the pieces together to accomplish that. We urge states to participate in MAP, to involve local school districts and to provide necessary funding, time and other assistance to make it work."

Sandra Feldman
President, American Federation of Teachers